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Suppose that $f\colon U \to \mathbb{C}$ is a meromorphic and has a zero of order k or a pole of order k at $z_0 \in U$. Then $f'(z)/f(z)$ has a simple pole at z_0 with residue k or $-k$ respectively.

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Since $g(z) \neq 0$ near z_0, $g'(z)/g(z)$ is holomorphic near z_0 so the result follows. The case where f has a zero at z_0 is similar. $\square$

Remark

Note that if U is an open set on which one can define a holomorphic branch L of $[Log(z)]$ then $g(z) = L(f(z))$ has $g'(z) = f'(z)/f(z)$.

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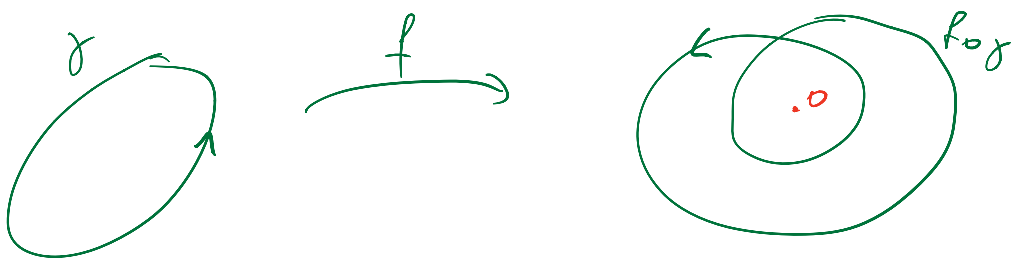

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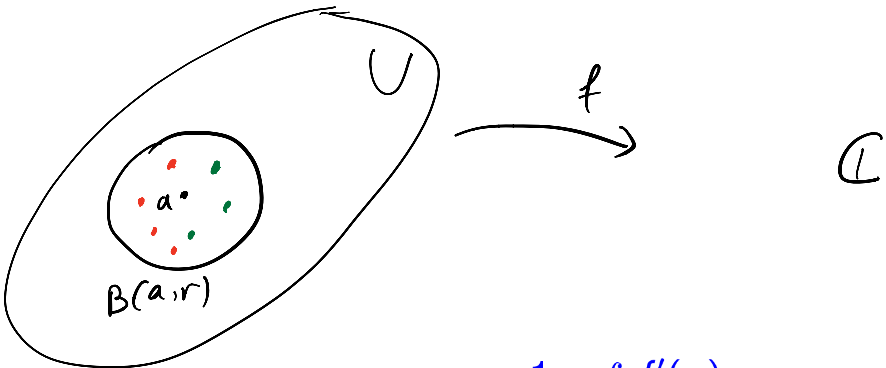
We will show using the residue theorem how to relate this to the number of zeros and poles of f inside γ:

Theorem

(Argument principle): Suppose that U is an open set and $f\colon U \to \mathbb{C}$ is a meromorphic function on U. If $B(a,r) \subseteq U$ and N is the number of zeros (counted with multiplicity) and P is the number of poles (again counted with multiplicity) of f inside $B(a,r)$ and f has neither on $\partial B(a,r)$ then

$$N - P = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}dz,$$

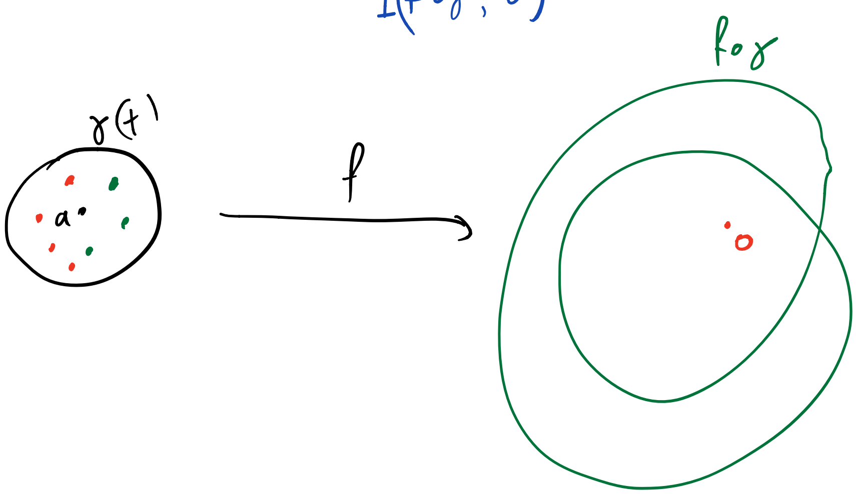
where $\gamma(t) = a + re^{2\pi it}$ is a path with image $\partial B(a,r)$. Moreover this is the winding number of the path $\Gamma = f \circ \gamma$ about the origin.



$$N - P = \text{zeros} - \text{poles} = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz,$$

$$\parallel$$

$$I(f\circ\gamma, 0)$$



$I(f\circ\gamma, 0)$ = change of the argument from $f(\gamma(0))$ to $f(\gamma(1))$.

Proof.

Clearly $I(\gamma, z)$ is 1 if $|z-a| \leq 1$ and is 0 otherwise.

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Recall that by the residue theorem

$$\frac{1}{2\pi i}\int_{\gamma} g(z)dz = \sum_{z_0 \in S} \mathrm{Res}_{z_0}(g) \cdot I(\gamma, z_0),$$

where the sum ranges over the poles z_0 of g inside γ.

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where the sum ranges over the poles z_0 of g inside γ.

By the previous lemma $f'(z)/f(z)$ has simple poles exactly at the zeros and poles of f with residues the corresponding orders. So the result follows (take $g(z) = f'(z)/f(z)$).

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For the last part, note that $2\pi i \cdot I(f \circ \gamma, 0)$ is just

$$\int_{f\circ\gamma} dw/w = \int_0^1 \frac{1}{f(\gamma(t))} f'(\gamma(t))\gamma'(t)dt = \int_\gamma \frac{f'(z)}{f(z)} dz.$$

$\square$

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The argument principle also holds, with the same proof, to any closed path γ *on which f is continuous and non-vanishing, provided it has winding number* $+1$ *around its inside.*

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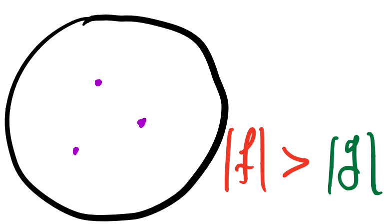
Theorem

(Rouché's theorem): Suppose that f and g are holomorphic functions on an open set U in $\mathbb{C}$ and $\bar{B}(a,r) \subset U$. If $|f(z)| > |g(z)|$ for all $z \in \partial B(a,r)$ then f and $f+g$ have the same number of zeros in $B(a,r)$ (counted with multiplicities).



Proof.

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Consider $h = (f+g)/f = 1 + g/f$. By hypothesis

$$|h(z) - 1| = |g(z)/f(z)| < 1$$

for all $z \in \gamma^*$.

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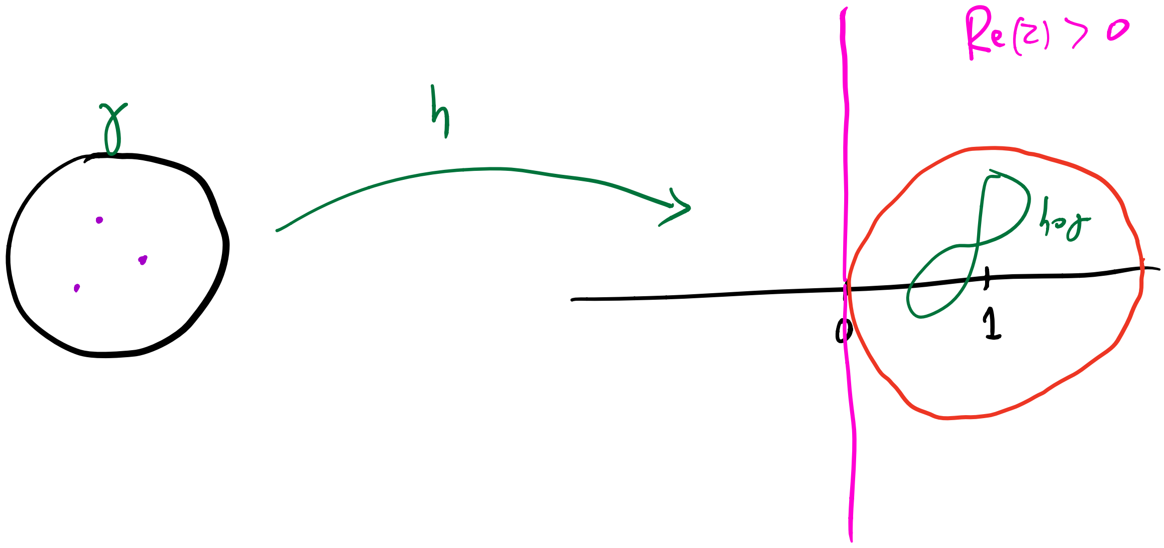
So $\Gamma(t) = h(\gamma(t))$ is contained in the half-plane $\{z : \Re(z) > 0\}$.



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By the argument principle $h = (f + g)/f$ has the same number of zeros as poles. As the number of poles is the number of zeros of f and the number of zeros is the number of zeros of $f + g$ the theorem follows. $\square$

Remark

Rouché's theorem can be useful in counting the number of zeros of a function f – one tries to find an approximation to f whose zeros are easier to count and then by Rouché's theorem obtain information about the zeros of f.
Just as for the argument principle above, Rouché's theorem also holds for closed paths which winding number 1 about their inside.

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(apply to $P-g$, $g \Rightarrow P-g$, P)

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As 0 has multiplicity 4 for $P - g$, the four roots of $P(z)$ all have modulus less than 2.
We note further that if we take $|z| = 1$, then $|5z + 2| \geq 5 - 2 = 3 > |z^4| = 1$, hence $P(z)$ and $5z + 2$ have the same number of roots in $B(0, 1)$. It follows $P(z)$ has one root of modulus less than 1, and 3 of modulus between 1 and 2.

apply to $g, P-g \Rightarrow g, P$

Open Mapping Theorem

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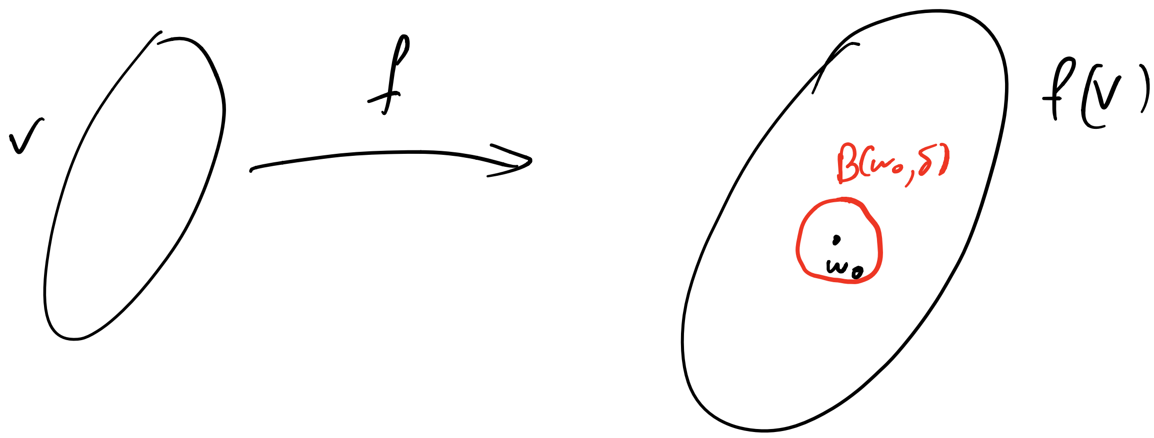

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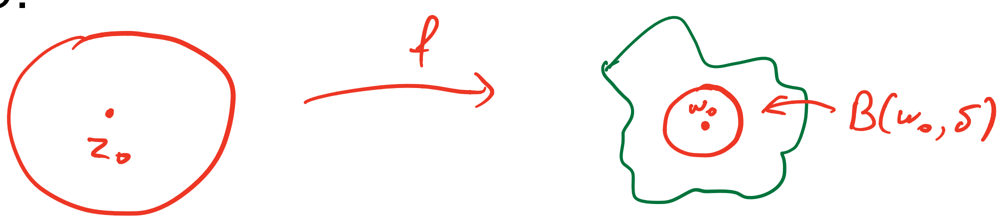
Thus we may find an $r > 0$ such that $g(z) \neq 0$ on $\bar{B}(z_0, r) \backslash \{z_0\} \subset U$.



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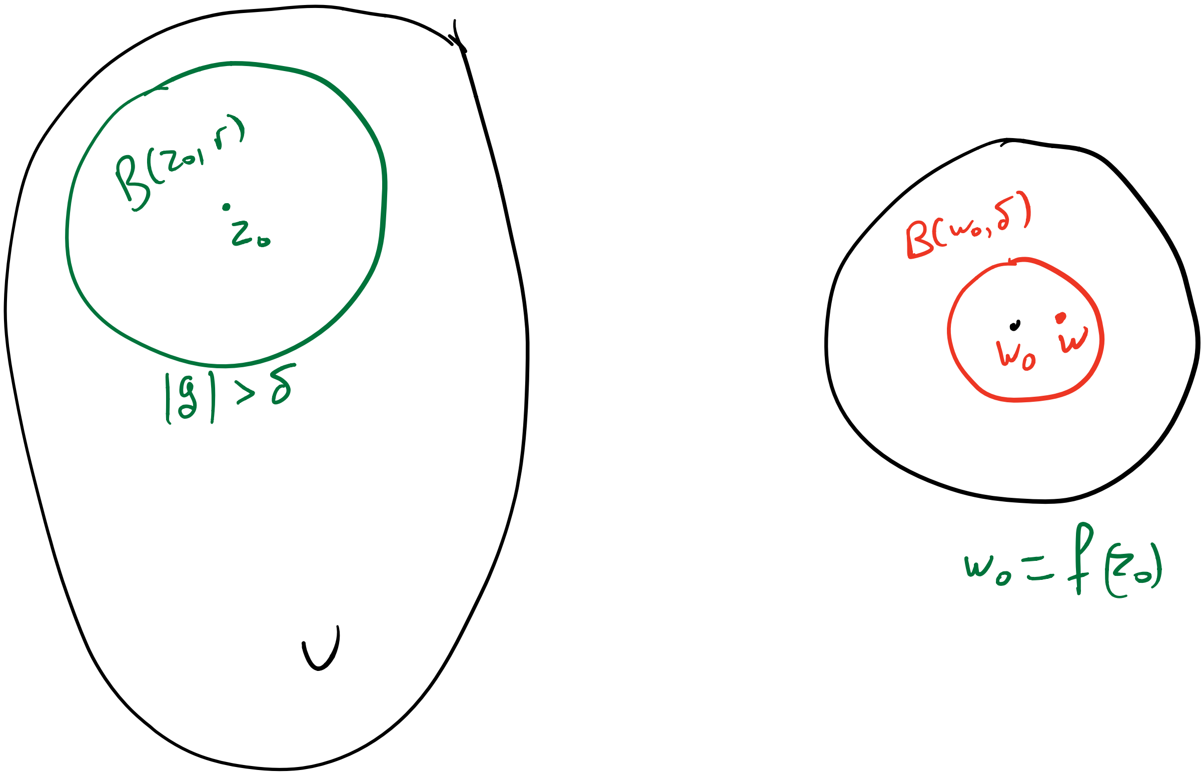
Since $\partial B(z_0, r)$ is compact, we have $|g(z)| \geq \delta > 0$ on $\partial B(z_0, r)$.

But then if $|w - w_0| < \delta$ it follows $|w - w_0| < |g(z)|$ on $\partial B(z_0, r)$.



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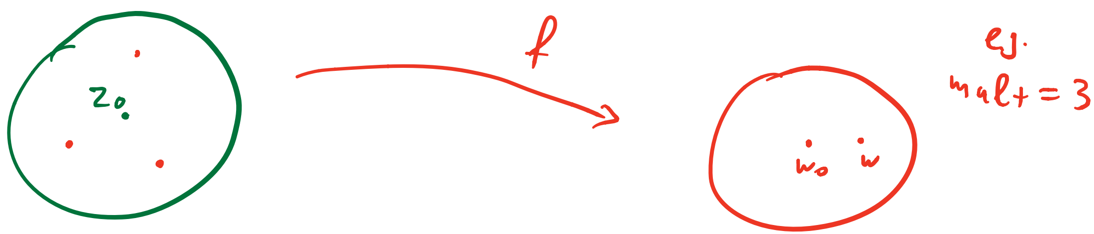

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We showed that $f(z) - w$ has as many zeros as $f(z) - w_0$ so f is locally d-to-1, counting multiplicities, that is, there are $r, \delta \in \mathbb{R}_{>0}$ such that for every $w \in B(w_0, \delta)$ the equation $f(z) = w$ has d solutions counted with multiplicity in the disk $B(z_0, r)$.

Inverse function theorem

Theorem

(Inverse function theorem): Suppose that $f\colon U \to \mathbb{C}$ *is injective and holomorphic and that* $f'(z) \neq 0$ *for all* $z \in U$*. If* $g\colon f(U) \to U$ *is the inverse of* f*, then* g *is holomorphic with* $g'(w) = 1/f'(g(w))$*.*

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g is holomorphic: fix $w_0 \in f(U)$ and let $z_0 = g(w_0)$. Note that since g and f are continuous, if $w \to w_0$ then $g(w) \to z_0$. Writing $z = f(w)$ we have

$$\lim_{w \to w_0} \frac{g(w) - g(w_0)}{w - w_0} = \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)} = 1/f'(z_0)$$

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But then z_0 is a root of multiplicity k of $f(z) - f(z_0) = 0$ so $f(z)$ is locally k-to-1 near z_0.

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Recall that if a is an isolated singularity of f and

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n, \quad \forall z \in B(a,r) \backslash \{a\}.$$

then the residue $\mathrm{Res}_a(f)$ of f at a is c_{-1} and

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is the principal part of f at a. $P_a(f)$ is holomorphic on $\mathbb{C} \setminus \{a\}$

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It turns out that it is possible to use this method and calculate ordinary integrals of real functions. There are several tricks that allow us to pass from an integral of a real function to a path integral of a complex function.

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Theorem

(Residue theorem): Suppose that U is an open set in $\mathbb{C}$ and γ is a closed path whose inside is contained in U, so that for all $z \notin U$ we have $I(\gamma, z) = 0$. Then if $S \subset U$ is a finite set such that $S \cap \gamma^* = \emptyset$ and f is a holomorphic function on $U \backslash S$ we have

$$\frac{1}{2\pi i} \int_{\gamma} f(z)dz = \sum_{a \in S} I(\gamma, a) Res_a(f)$$

Proof.

For each $a \in S$ let $P_a(f)(z) = \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n$ be the principal part of f at a, a holomorphic function on $\mathbb{C} \backslash \{a\}$.

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Then $f - P_a(f)$ is holomorphic at $a \in S$, and thus $g(z) = f(z) - \sum_{a \in S} P_a(f)$ is holomorphic on all of U.

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Then $f - P_a(f)$ is holomorphic at $a \in S$, and thus $g(z) = f(z) - \sum_{a \in S} P_a(f)$ is holomorphic on all of U.

So by Cauchy's Theorem $\int_\gamma g(z)dz = 0$, hence

$$\int_\gamma f(z)dz = \sum_{a \in S} \int_\gamma P_a(f)(z)dz$$

Proof.

For each $a \in S$ let $P_a(f)(z) = \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n$ be the principal part of f at a, a holomorphic function on $\mathbb{C}\backslash\{a\}$.

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But the series $P_a(f)$ converges uniformly on γ^* so that

$$\int_\gamma P_a(f)dz = \int_\gamma \sum_{n=-1}^{-\infty} c_n(a)(z-a)^n = \sum_{n=1}^{\infty} \int_\gamma \frac{c_{-n}(a)dz}{(z-a)^n}$$
$$= \int_\gamma \frac{c_{-1}(a)dz}{z-a} = 2\pi i \cdot I(\gamma, a)\mathrm{Res}_a(f),$$

since for $n > 1$ the function $(z-a)^{-n}$ has a primitive on $\mathbb{C}\backslash\{a\}$.

Remark

In applications the winding numbers $I(\gamma, a)$ will be simple to compute in terms of the argument of $(z-a)$ - in fact most often they will be 0 or ± 1 as we will usually apply the theorem to integrals around some standard contours that are simple closed curves.

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$$\int_0^{2\pi} \frac{dt}{1+3\cos^2(t)} = \int_\gamma \frac{-4iz}{3 + 10z^2 + 3z^4} dz.$$

Thus we have turned our real integral into a contour integral, and to evaluate the contour integral we just need to calculate the residues of the meromorphic function $g(z) = \frac{-4iz}{3+10z^2+3z^4}$ at the poles it has inside the unit circle.

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The poles of $g(z)$ are the zeros of $p(z) = 3 + 10z^2 + 3z^4$, which are at $z^2 \in \{-3, -1/3\}$. Thus the poles inside the unit circle are at $\pm i/\sqrt{3}$.

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Since p has degree 4 and has four roots, they must all be simple zeros, and so g has simple poles at these points.

The residue at a simple pole z_0 can be calculated as the limit $\lim_{z \to z_0} (z - z_0) g(z)$, thus

$$\mathrm{Res}_{z=\pm i/\sqrt{3}}(g(z)) = \lim_{z \to \pm i/\sqrt{3}} \frac{-4iz(z - \pm i/\sqrt{3})}{3 + 10z^2 + 3z^4}$$

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It now follows from the Residue theorem that

$$\int_0^{2\pi} \frac{dt}{1+3\cos^2(t)} = 2\pi i\left(\mathrm{Res}_{z=i/\sqrt{3}}((g(z)) + \mathrm{Res}_{z=-i/\sqrt{3}}(g(z))\right) = \pi.$$

Applications of The Residue Theorem

Theorem

(Residue theorem): Suppose that U is an open set in $\mathbb{C}$ and γ is a path whose inside is contained in U, so that for all $z \notin U$ we have $I(\gamma, z) = 0$. Then if $S \subset U$ is a finite set such that $S \cap \gamma^* = \emptyset$ and f is a holomorphic function on $U \backslash S$ we have

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in S} I(\gamma, a) Res_a(f)$$

Remark

Often we are interested in integrating along a path which is not closed or even finite, for example, we might wish to understand the integral of a function on the positive real axis.

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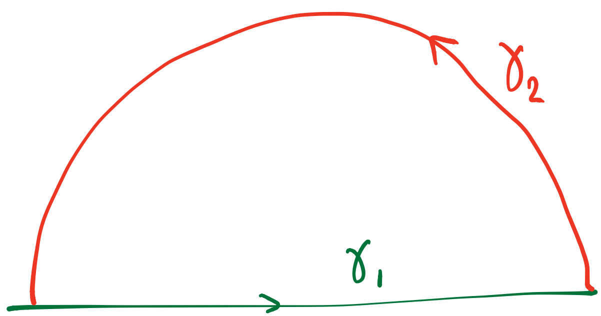
Often we are interested in integrating along a path which is not closed or even finite, for example, we might wish to understand the integral of a function on the positive real axis.

The residue theorem can still be a powerful tool in calculating these integrals, provided we complete the path to a closed one in such a way that we can control the extra contribution to the integral along the part of the path we add.

If we have a function f which we wish to integrate over the whole real line (so we have to treat it as an improper Riemann integral) then we may consider the contours Γ_R given as the concatenation of the paths $\gamma_1 : [-R, R] \to \mathbb{C}$ and $\gamma_2 : [0, 1] \to \mathbb{C}$ where

$$\gamma_1(t) = -R + t; \quad \gamma_2(t) = Re^{i\pi t}.$$

(so that $\Gamma_R = \gamma_2 \star \gamma_1$ traces out the boundary of a half-disk).



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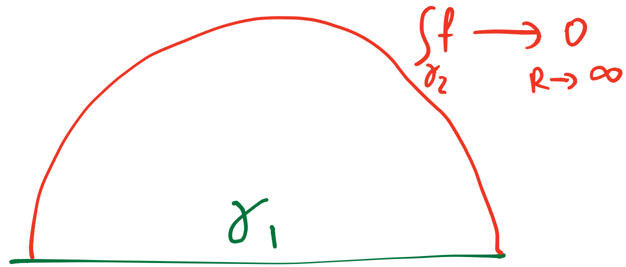
In many cases one can show that $\int_{\gamma_2} f(z)dz$ tends to 0 as $R \to \infty$, and by calculating the residues inside the contours Γ_R deduce the integral of f on $(-\infty, \infty)$.



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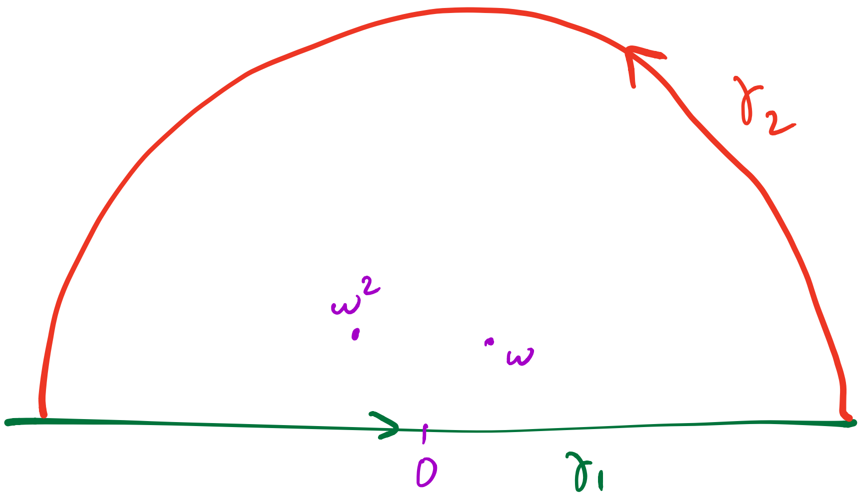
The function $f(z) = 1/(1+z^2+z^4)$ has poles at $z^2 = \pm e^{2\pi i/3}$ and hence at $\{e^{\pi i/3}, e^{2\pi i/3}, e^{4\pi i/3}, e^{5\pi i/3}\}$. They are all simple poles and of these only $\{\omega, \omega^2\}$ are in the upper-half plane, where $\omega = e^{i\pi/3}$.

Thus by the residue theorem, for all $R > 1$ we have

$$\int_{\Gamma_R} f(z)dz = 2\pi i\big(\mathrm{Res}_{\omega}(f(z)) + \mathrm{Res}_{\omega^2}(f(z))\big),$$



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We calculate the residues:

$$\mathrm{Res}_{\omega}(f(z)) = \lim_{z\to\omega} \frac{(z-\omega)}{1+z^2+z^4} = \frac{1}{2\omega+4\omega^3} = \frac{1}{2\omega-4}$$

$$\mathrm{Res}_{\omega^2}(f(z)) = \frac{1}{2\omega^2+4\omega^6} = \frac{1}{4+2\omega^2}$$

$p(z) = (z-\omega)\, q(z) \Rightarrow p'(z) = q(z) + (z-\omega)\, q'(z) \Rightarrow$

$\Rightarrow q(\omega) = p'(\omega)$

$\omega^3 = 1$

$p'(z) = 2z + 4z^3$

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$$\int_{\Gamma_R} f(z)dz = 2\pi i\Big(\frac{1}{2\omega-4} + \frac{1}{2\omega^2+4}\Big) = \pi i\Big(\frac{1}{\omega-2} + \frac{1}{\omega^2+2}\Big)$$

$$= \pi i\Big(\frac{\omega^2+\omega}{2(\omega-\omega^2)-5}\Big) = -\sqrt{3}\pi/(-3) = \pi/\sqrt{3},$$

(where we used the fact that $\omega^2+\omega = i\sqrt{3}$ and $\omega-\omega^2 = 1$).

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so we need to calculate the limit of $\int_{\gamma_2} f(z)dz$ as $R \to \infty$. By the estimation lemma we have

$$\left|\int_{\gamma_2} f(z)dz\right| \leq \sup_{z\in\gamma_2^*} |f(z)| \cdot \ell(\gamma_2) \leq \frac{\pi R}{R^4 - R^2 - 1} \to 0,$$

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hence

$$\pi/\sqrt{3} = \lim_{R\to\infty} \int_{\Gamma_R} f(z)dz = \int_{-\infty}^{\infty} \frac{dt}{1+t^2+t^4}.$$

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Theorem

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Jordan's Lemma and applications

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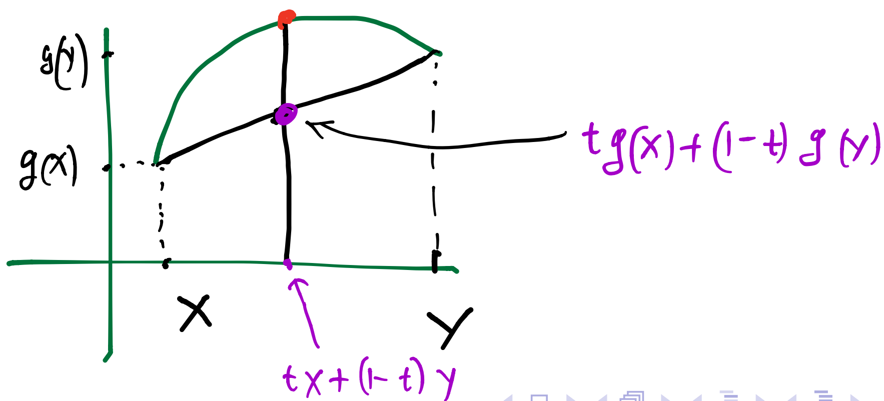
Recall:

Lemma

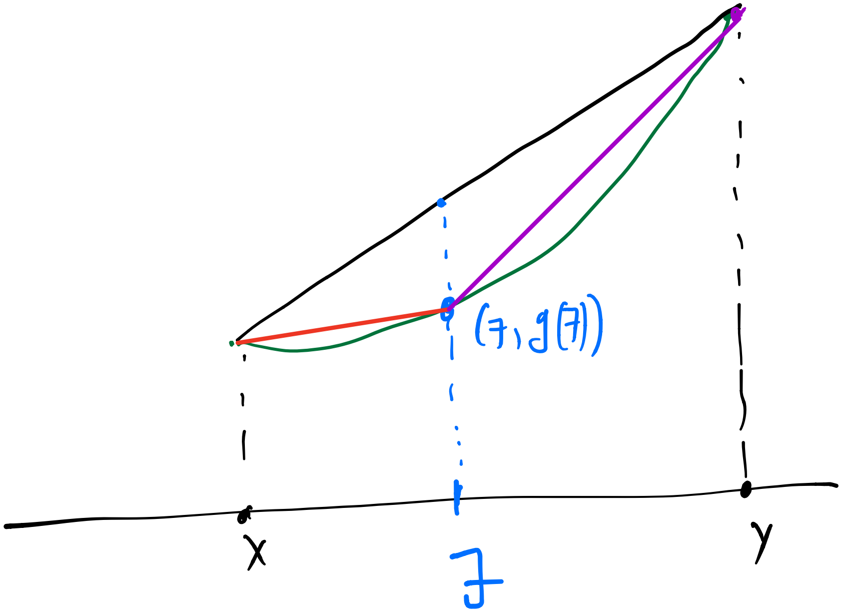
Let $g\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function. Then if $[a, b]$ is an interval on which $g''(x) < 0$, the function g is concave on $[a, b]$, that is, for $x < y \in [a, b]$ we have

$$g(tx + (1 - t)y) \geq tg(x) + (1 - t)g(y), \quad t \in [0, 1].$$



Proof.

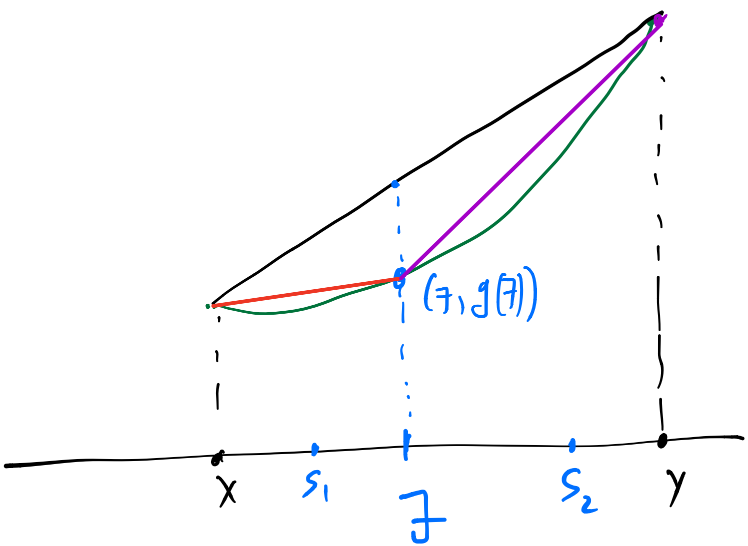
Given $x, y \in [a, b]$ and $t \in [0, 1]$ let $\xi = tx + (1 - t)y$, a point in the interval between x and y.



Proof.

Given $x, y \in [a, b]$ and $t \in [0, 1]$ let $\xi = tx + (1 - t)y$, a point in the interval between x and y.

The slope of the chord between $(x, g(x))$ and $(\xi, g(\xi))$ is, by the Mean Value Theorem, equal to $g'(s_1)$ where s_1 lies between x and ξ, while the slope of the chord between $(\xi, g(\xi))$ and $(y, g(y))$ is equal to $g'(s_2)$ for s_2 between ξ and y.



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If $\underline{g(\xi) < tg(x) + (1 - t)g(y)}$ it follows that $g'(s_1) < \dfrac{g(y) - g(x)}{y - x}$ and $g'(s_2) > \dfrac{g(y) - g(x)}{y - x}$.

$$g'(s_1) = \frac{g(\xi) - g(x)}{\xi - x} < \frac{t\,g(x) + (1-t)\,g(y) - g(x)}{t x + (1-t) y - x} = \frac{(\cancel{1-t})\,[g(y) - g(x)]}{(\cancel{1-t})\,(y - x)}$$

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If $g(\xi) < tg(x) + (1-t)g(y)$ it follows that $g'(s_1) < \dfrac{g(y) - g(x)}{y - x}$ and $g'(s_2) > \dfrac{g(y) - g(x)}{y - x}$.

Thus by the mean value theorem for $g'(x)$ applied to the points s_1 and s_2 it follows there is an $s \in (s_1, s_2)$ with $g''(s) = (g'(s_2) - g'(s_1))/(s_2 - s_1) > 0$, contradicting the assumption that $g''(x)$ is negative on (a, b). $\square$

Corollary

$\sin(t) \geq \frac{2}{\pi} t$ *for* $t \in [0, \pi/2]$ *and* $\sin(\pi - t) \geq 2(\pi - t)/\pi$ *for* $t \in [\pi/2, \pi]$.

Corollary

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Proof.

By the lemma since $\sin'' t < 0$ in $(0, \pi/2)$

$$\sin t = \sin\left((1 - \frac{2}{\pi}t) \cdot 0 + \frac{2}{\pi}t \cdot \frac{\pi}{2}\right) \geq (1 - \frac{2}{\pi}t) \sin 0 + \frac{2}{\pi}t \sin\frac{\pi}{2} = \frac{2}{\pi}t.$$

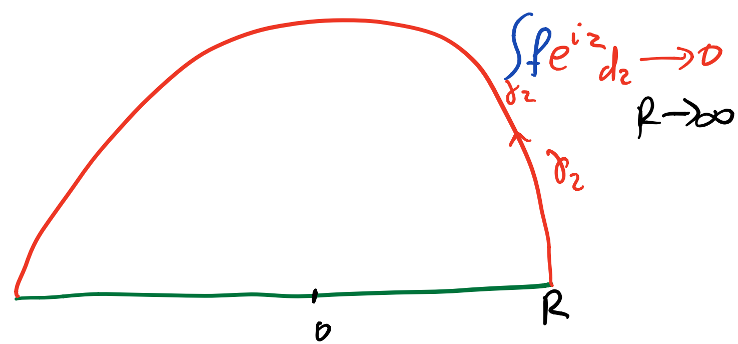
Clearly for $t \in [\pi/2, \pi]$, $\pi - t \in [0, \pi/2]$ so the same inequality applies. □

Lemma

(Jordan's Lemma): Let $f : \mathbb{H} \to \mathbb{C}_\infty$ *be a meromorphic function on the upper-half plane* $\mathbb{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$*. Suppose that* $f(z) \to 0$ *as* $z \to \infty$ *in* $\mathbb{H}$*. Then if* $\gamma_R(t) = Re^{it}$ *for* $t \in [0, \pi]$ *we have*

$$\int_{\gamma_R} f(z)e^{i\alpha z}dz \to 0$$

as $R \to \infty$ *for all* $\alpha \in \mathbb{R}_{>0}$.



Proof.

Suppose that $\epsilon > 0$ is given. Then by assumption we may find an S such that for $|z| > S$ we have $|f(z)| < \epsilon$. Thus if $R > S$ and $z = \gamma_R(t)$, it follows that

$$|f(z)e^{i\alpha z}| \leq \epsilon e^{-\alpha R \sin(t)}.$$

$$z = Re^{it}, \quad e^{i\alpha z} = e^{i\alpha R(\cos t + i \sin t)} =$$

$$= e^{iR\cos t} e^{-\alpha R \sin(t)}$$

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$$|f(z)e^{i\alpha z}| \quad \leq \epsilon e^{-\alpha R \sin(t)}.$$

By the corollary we have

$$|f(z)e^{i\alpha z}| \leq \begin{cases} \epsilon \cdot e^{-2\alpha Rt/\pi}, & t \in [0, \pi/2] \\ \epsilon \cdot e^{-2\alpha R(\pi - t)/\pi} & t \in [\pi/2, \pi] \end{cases}$$

$$\sin(t) \geq \frac{2}{\pi} t \Rightarrow e^{-\sin t} \leq e^{-\frac{2}{t}\pi}$$

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But then it follows that

$$\left|\int_{\gamma_R} f(z)e^{i\alpha z}dz\right| \leq 2\int_0^{\pi/2} \epsilon R \cdot e^{-2\alpha Rt/\pi}dt = \epsilon \cdot \pi \frac{1 - e^{-\alpha R}}{\alpha} < \epsilon \cdot \pi/\alpha,$$

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But π/α is constant, so $\int_{\gamma_R} f(z)e^{i\alpha z} dz \to 0$ as $R \to \infty$ $\square$

Remark

If η_R is an arc of a semicircle in the upper half plane, say $\eta_R(t) = Re^{it}$ for $0 \leq t \leq 2\pi/3$, then the same proof shows that

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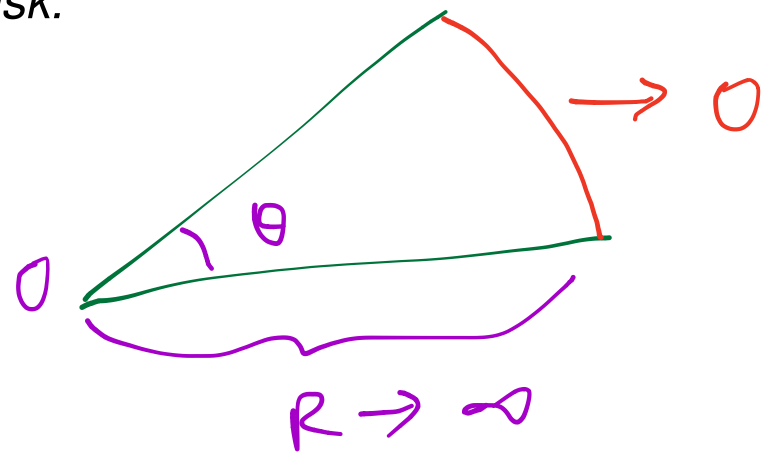
This is sometimes useful when integrating around the boundary of a sector of disk.



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Note that if $\alpha < 0$ then the integral of $f(z)e^{i\alpha z}$ around a semicircle in the lower half plane tends to zero as $R \to \infty$ provided $|f(z)| \to 0$ as $|z| \to \infty$ in the lower half plane. This follows immediately from the above applied to $f(-z)$.

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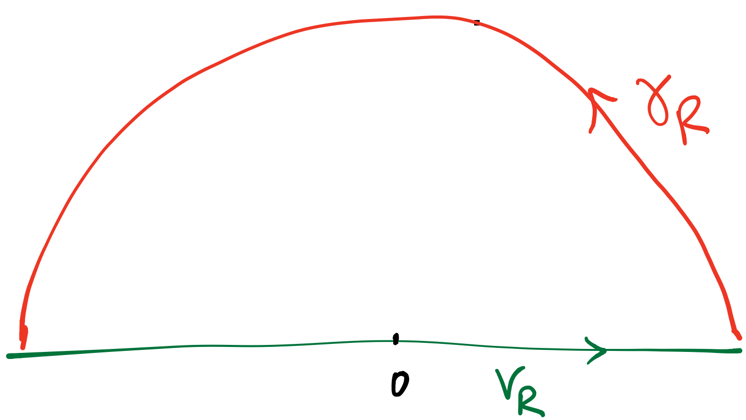

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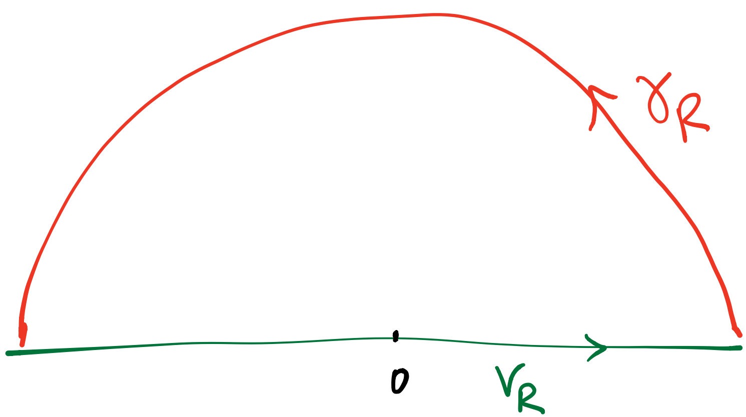

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Thus we have $\int_{\eta_R} f(z) dz = 0$ for all $R > 0$.

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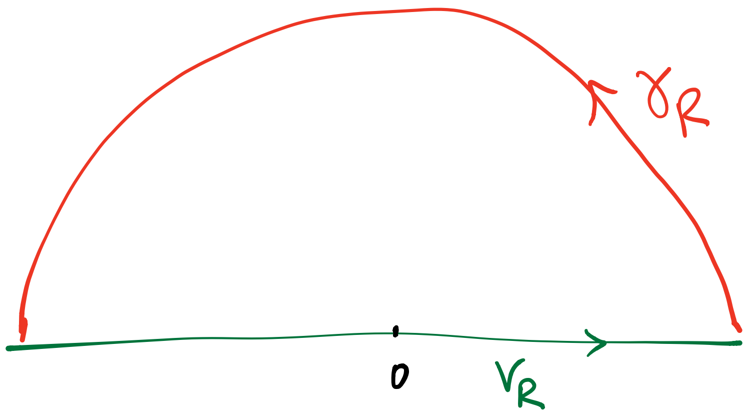

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$f(t) = \dfrac{\cos t + i \sin t}{t}$ so

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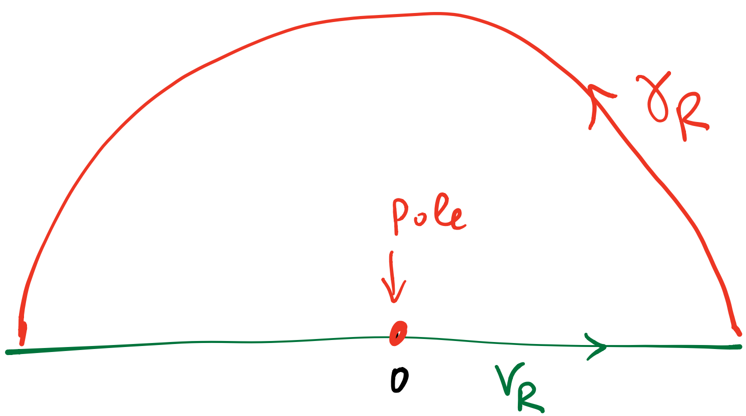

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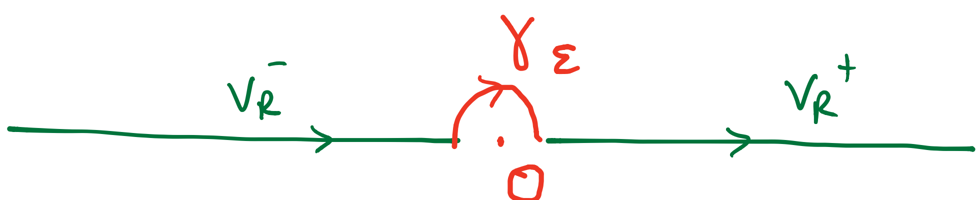

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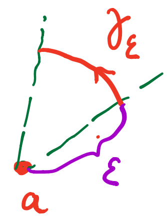
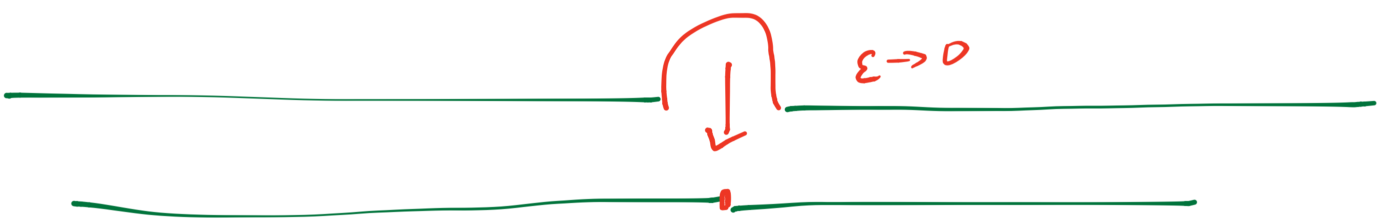
How can we calculate the value of the integral after this change? We have a general lemma:

Lemma

Let $f\colon U \to \mathbb{C}$ be a meromorphic function with a simple pole at $a \in U$ and let $\gamma_\epsilon\colon [\alpha, \beta] \to \mathbb{C}$ be the path $\gamma_\epsilon(t) = a + \epsilon e^{it}$, then

$$\lim_{\epsilon \to 0} \int_{\gamma_\epsilon} f(z)dz = Res_a(f) \cdot (\beta - \alpha)i.$$





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Proof.

Since f has a simple pole at a, we may write

$$f(z) = \frac{c}{z-a} + g(z)$$

where $g(z)$ is holomorphic near z and $c = \mathrm{Res}_a(f)$.

As g is holomorphic at a, it is continuous at a, and so bounded. Let $M, r > 0$ be such that $|g(z)| < M$ for all $z \in B(a, r)$. Then if $0 < \epsilon < r$ we have

$$|\int_{\gamma_\epsilon} g(z)dz| \leq \ell(\gamma_\epsilon)M = (\beta - \alpha)\epsilon \cdot M \to 0$$

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Since $\int_{\gamma_\epsilon} f(z)dz = \int_{\gamma_\epsilon} c/(z-a)dz + \int_{\gamma_\epsilon} g(z)dz$ the result follows. $\square$

We return now to the calculation of the integral $\int_{-\infty}^{\infty} \frac{\sin(x)}{x} dx$ using the more ‘obvious’ function $\dfrac{e^{iz}}{z}$.

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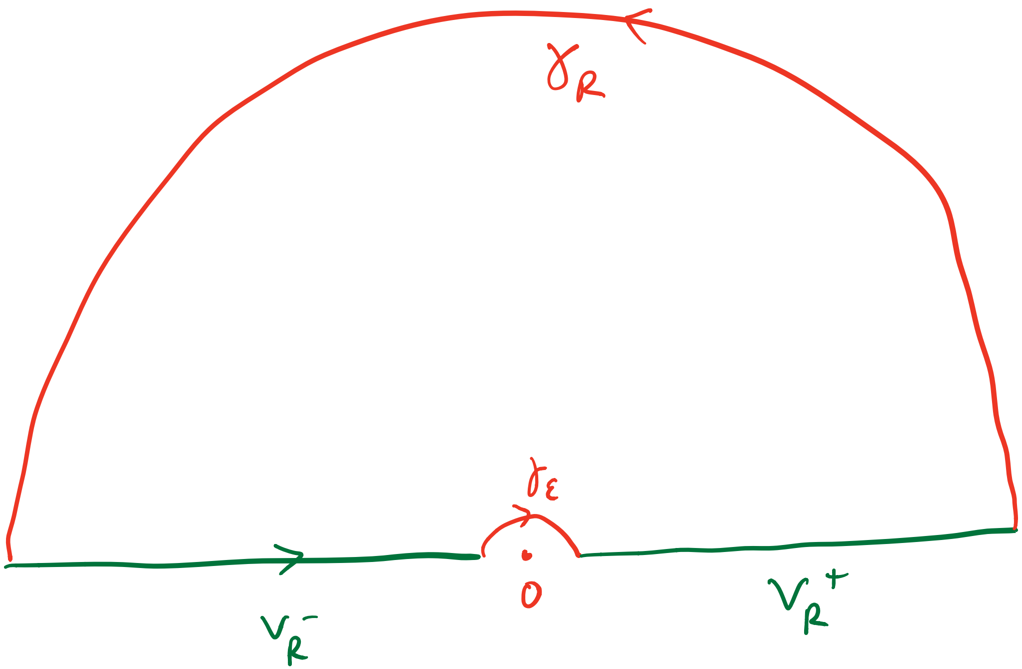
as $\epsilon \to 0$.

Integrating then $\dfrac{e^{iz}}{z}$ over $\Gamma_\epsilon = \nu_R^- \star \gamma_\epsilon \star \nu_R^+ \star \gamma_R$, we get:



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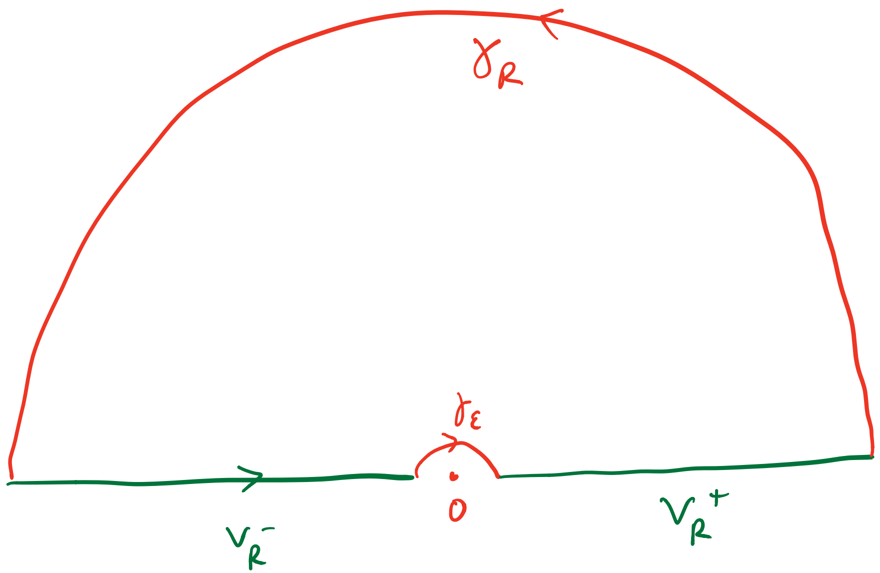

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$$= 2i\int_{\epsilon}^{R} \frac{\sin(x)}{x} + \int_{\gamma_\epsilon} \frac{e^{iz}}{z} + \int_{\gamma_R} \frac{e^{iz}}{z}dz$$

use $\lim_{\epsilon\to 0} \int_{\gamma_\epsilon} f(z)dz = Res_a(f) \cdot (\beta - \alpha)i.$

here $Res_0 = 1$ $\beta = 0$, $\alpha = \pi$ for γ_ϵ

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Then letting $R \to \infty$, it follows from Jordan's Lemma that the third term tends to zero so we see that

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx = 2\int_{0}^{\infty} \frac{\sin(x)}{x}dx = \pi$$

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How do we calculate these?

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We discuss now a more direct method to calculate the residue in the case of functions which are given as the ratio of two holomorphic functions.

Precisely let $F \colon U \to \mathbb{C}$ given to us as a ratio f/g of two holomorphic functions f, g on U. The singularities of the function F are therefore poles which are located precisely at the (isolated) zeros of the function g.

For convenience, we assume that we have translated the plane so as to ensure the pole of $F = f/g$ we are interested in is at $a = 0$.

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We set $h(z) = \sum_{n=1}^{\infty} a_n z^{n-1}$, then

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We can 'ignore' the terms after k as:

$$\sum_{m \geq k} (-1)^m z^m h(z)^m = z^k h_1(z)$$

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Hence the principal part of the Laurent series of $1/g(z)$ is equal to the principal part of the function

$$\frac{1}{c_k z^k} \sum_{n=1}^{k} (-1)^{k-1} z^k h(z)^k$$

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Finally, the principal part $P_0(F)$ of $F = f/g$ at $z = 0$ is just the principal part of the function $f(z) \cdot P_0(g)$, which again we can compute if we know the power series expansion of $f(z)$ at 0.

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1) $e^{x+iy} - e^{-x-iy} = 0 \Rightarrow e^x = e^{-x} \Rightarrow x = 0$

and $e^{2iy} = 1 \Rightarrow y \in \pi i \mathbb{Z}$

.

2) $f(z) = (z-a)^2 g(z) \Rightarrow f'(a) = 0$

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Thus $f(z)$ has a pole or order 5 at zero, and poles of order 3 at πin for each $n \in \mathbb{Z}\backslash\{0\}$. We calculate the principal part of f at $z = 0$.

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Thus $f(z)$ has a pole or order 5 at zero, and poles of order 3 at πin for each $n \in \mathbb{Z}\backslash\{0\}$. We calculate the principal part of f at $z = 0$.

We will write $O(z^k)$ for holomorphic functions which have a zero of order at least k at 0.

$$z^2 \sinh(z)^3 = z^2(z + \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7))^{\color{red}3} = z^5(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6))^{\color{red}3}$$

$$z^2 \sinh(z)^3 = z^2\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)\right)^{3} = z^5\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6)\right)^{3}$$

$$= z^5\left(1 + \frac{3z^2}{3!} + \frac{3z^4}{(3!)^2} + \frac{3z^4}{5!} + O(z^6)\right)$$

$$\left(1 + \frac{z^2}{3!}\right)^3 = 1 + \frac{3z^2}{3!} + \frac{3z^4}{(3!)^2} + O(z^6)$$

$$
\begin{aligned}
z^2\sinh(z)^3 &= z^2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+O(z^7)\right)^{3} = z^5\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+O(z^6)\right)^{3}\\
&= z^5\left(1+\frac{3z^2}{3!}+\frac{3z^4}{(3!)^2}+\frac{3z^4}{5!}+O(z^6)\right)\\
&= z^5\left(1+\frac{z^2}{2}+\frac{13z^4}{120}+O(z^6)\right)
\end{aligned}
$$

$$
\begin{aligned}
z^2\sinh(z)^3 &= z^2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+O(z^7)\right)^{3} = z^5\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+O(z^6)\right)^{3} \\
&= z^5\left(1+\frac{3z^2}{3!}+\frac{3z^4}{(3!)^2}+\frac{3z^4}{5!}+O(z^6)\right) \\
&= z^5\left(1+\frac{z^2}{2}+\frac{13z^4}{120}+O(z^6)\right) \\
&= z^5\left(1+z\left(\frac{z}{2}+\frac{13z^3}{120}+O(z^5)\right)\right)
\end{aligned}
$$

$$z^2\sinh(z)^3 = z^2\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+O(z^7)\right)^{3} = z^5\left(1+\frac{z^2}{3!}+\frac{z^4}{5!}+O(z^6)\right)^{3}$$
$$= z^5\left(1+\frac{3z^2}{3!}+\frac{3z^4}{(3!)^2}+\frac{3z^4}{5!}+O(z^6)\right)$$
$$= z^5\left(1+\frac{z^2}{2}+\frac{13z^4}{120}+O(z^6)\right)$$
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Using our previous notation, $h(z)=\frac{z}{2}+\frac{13z^3}{120}+O(z^5)$

$$z^2 \sinh(z)^3 = z^2\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + O(z^7)\right)^3 = z^5\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + O(z^6)\right)^3$$

$$= z^5\left(1 + \frac{3z^2}{3!} + \frac{3z^4}{(3!)^2} + \frac{3z^4}{5!} + O(z^6)\right)$$

$$= z^5\left(1 + \frac{z^2}{2} + \frac{13z^4}{120} + O(z^6)\right)$$

$$= z^5\left(1 + z\left(\frac{z}{2} + \frac{13z^3}{120} + O(z^5)\right)\right)$$

Using our previous notation, $h(z) = \frac{z}{2} + \frac{13z^3}{120} + O(z^5)$

so to find the principal part we just need to consider the first two terms in the series $(1 + zh(z))^{-1} = \sum_{n=0}^{\infty}(-1)^n z^n h(z)^n$:

3rd term: $z^3 \cdot \left(\frac{z}{2} + \cdots\right)^3 = O(z^6)$

$$1/z^2 \sinh(z)^3 = z^{-5}\left(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\right)^{-1}$$

$$
\begin{aligned}
1/z^2\sinh(z)^3 &= z^{-5}\left(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\right)^{-1} \\
&= z^{-5}\left(1 - z(\frac{z}{2} + \frac{13z^3}{120}) + z^2\frac{z^2}{(2!)^2} + O(z^5)\right)
\end{aligned}
$$

$$
\begin{aligned}
1/z^2 \sinh(z)^3 &= z^{-5}\Big(1 + z(\frac{z}{2} + \frac{13z^3}{120} + O(z^5))\Big)^{-1} \\
&= z^{-5}\Big(1 - z(\frac{z}{2} + \frac{13z^3}{120}) + z^2\frac{z^2}{(2!)^2} + O(z^5)\Big) \\
&= z^{-5}\Big(1 - \frac{z^2}{2} + (\frac{1}{4} - \frac{13}{120})z^4 + O(z^5)\Big)
\end{aligned}
$$

$$
\begin{aligned}
1/z^2 \sinh(z)^3 &= z^{-5}\Big(1 + z\big(\frac{z}{2} + \frac{13z^3}{120} + O(z^5)\big)\Big)^{-1} \\
&= z^{-5}\Big(1 - z\big(\frac{z}{2} + \frac{13z^3}{120}\big) + z^2\frac{z^2}{(2!)^2} + O(z^5)\Big) \\
&= z^{-5}\Big(1 - \frac{z^2}{2} + \big(\frac{1}{4} - \frac{13}{120}\big)z^4 + O(z^5)\Big) \\
&= \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z} + O(z).
\end{aligned}
$$

$$
\begin{aligned}
1/z^2 \sinh(z)^3 &= z^{-5}\left(1 + z\left(\frac{z}{2} + \frac{13z^3}{120} + O(z^5)\right)\right)^{-1} \\
&= z^{-5}\left(1 - z\left(\frac{z}{2} + \frac{13z^3}{120}\right) + z^2\frac{z^2}{(2!)^2} + O(z^5)\right) \\
&= z^{-5}\left(1 - \frac{z^2}{2} + \left(\frac{1}{4} - \frac{13}{120}\right)z^4 + O(z^5)\right) \\
&= \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z} + O(z).
\end{aligned}
$$

Thus $P_0(f) = \frac{1}{z^5} - \frac{1}{2z^3} + \frac{17}{120z}$, and $\mathrm{Res}_0(f) = 17/120$