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(We calculate the zeros of $\sin z$ using $\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$).

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(We calculate the zeros of $\sin z$ using $\sin(z) = \frac{e^{iz} - e^{-iz}}{2}$).

Since f is periodic with period 1, it suffices to calculate the principal part of f at $z = 0$.

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$$\frac{1}{\sin(z)} = \frac{1}{z}(1-zh(z))^{-1} = \frac{1}{z}\Big(1+\sum_{n\geq 1} z^n h(z)^n\Big) = \frac{1}{z}+h(z)+O(z^2).$$

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$\cos(z) = 1 + O(z^2)$ so the principal part of $\cot(z)$ is $1/z$. It follows that $\cot(\pi z)$ has a simple pole at each $n \in \mathbb{Z}$ with residue $1/\pi$.

We can also calculate further terms of the Laurent series of $\cot(z)$: As $h(z)$ actually vanishes at $z = 0$, the terms $h(z)^n z^n$ vanish to order $2n$.

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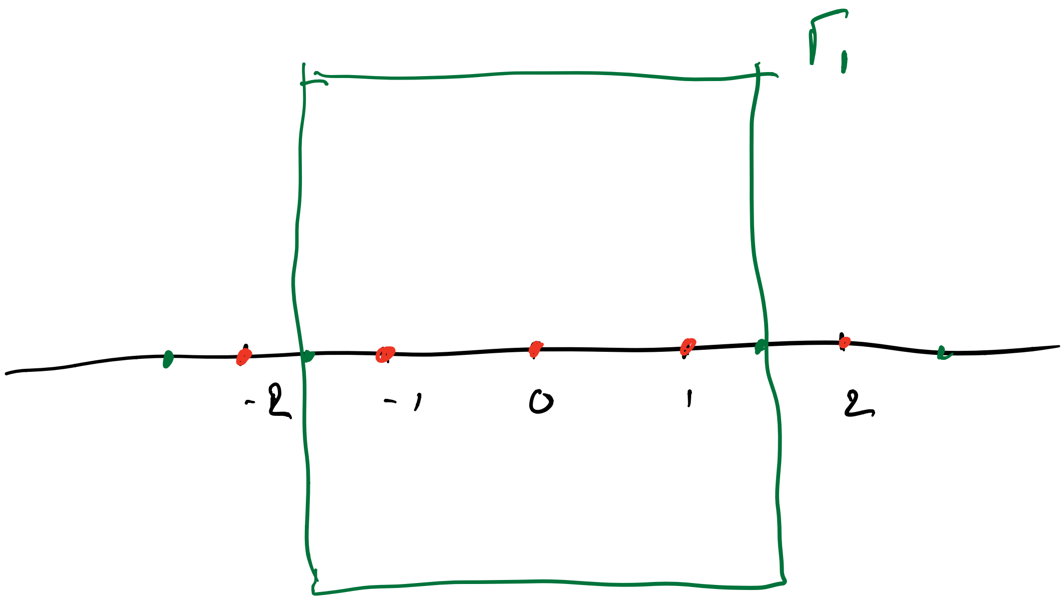
Since $\cos(z) = 1 - z^2/2! + O(z^4)$, it follows that $\cot(z)$ has a Laurent series

$$\begin{aligned}\cot(z) &= (1 - \frac{z^2}{2!} + O(z^4)) \cdot \Big(\frac{1}{z} + \frac{z}{3!} + O(z^3)\Big)\Big) \\ &= \frac{1}{z} - \frac{z}{3} + O(z^3)\end{aligned}$$

Lemma

Let $f(z) = \cot(\pi z)$ and let Γ_N denote the square path with vertices $(N + 1/2)(\pm 1 \pm i)$ where $N \in \mathbb{N}$. There is a constant C independent of N such that $|f(z)| \leq C$ for all $z \in \Gamma_N^*$.



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Proof.

Note that $\cot(\pi z) = (e^{i\pi z} + e^{-i\pi z})/(e^{i\pi z} - e^{-i\pi z})$.

Horizontal sides: $z = x \pm (N + 1/2)i$ and

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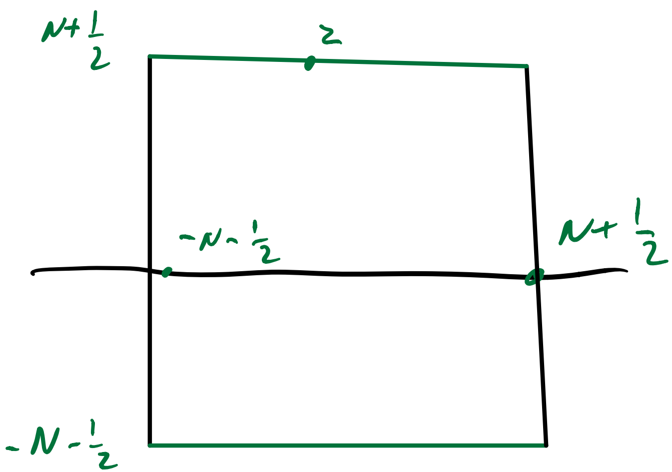

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as $|x + e^{i\theta} y| \le x + y$ for x, y positive reals and $|x - e^{i\theta} y| > x - y$.

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as e^{-x} is decreasing for $x > 0$.

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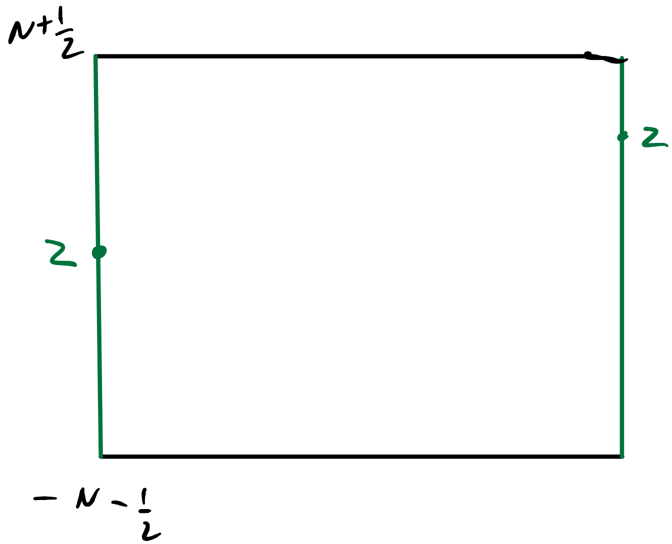

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so we can take $C = \dfrac{2}{1 - e^{-3\pi}}$. $\square$

Example Let $g(z) = \cot(\pi z)/z^2$. By the calculation of Laurent series of $\cot(\pi z)$ at $z = 0$:

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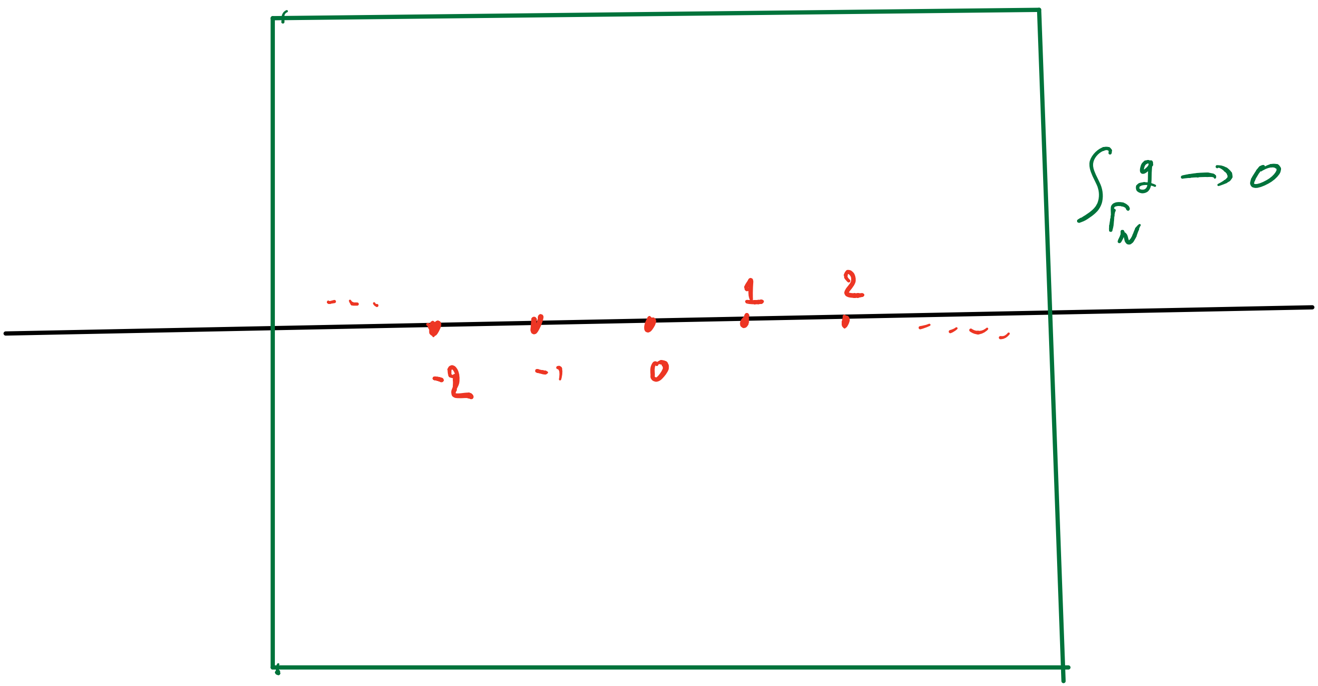
Consider now the integral of $g(z)$ around the paths Γ_N: We know $|g(z)| \leq C/|z|^2$ for $z \in \Gamma_N^*$, and for all $N \geq 1$. Thus by the estimation lemma

$$\left(\int_{\Gamma_N} g(z)dz\right) \leq C \cdot (4N + 2)/(N + 1/2)^2 \to 0,$$

as $N \to \infty$.

But by the residue theorem we know that

$$\int_{\Gamma_N} g(z)dz = -\pi/3 + \sum_{\substack{n \neq 0, \\ -N \leq n \leq N}} \frac{1}{\pi n^2}.$$



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Remark

Notice that the contours Γ_N and the function $\cot(\pi z)$ clearly allows us to sum other infinite series in a similar way – for example if we wished to calculate the sum of the infinite series $\sum_{n\geq 1} \frac{1}{n^2+1}$ *then we would consider the integrals of* $g(z) = \cot(\pi z)/(1+z^2)$ *over the contours Γ_N.*

Keyhole contours

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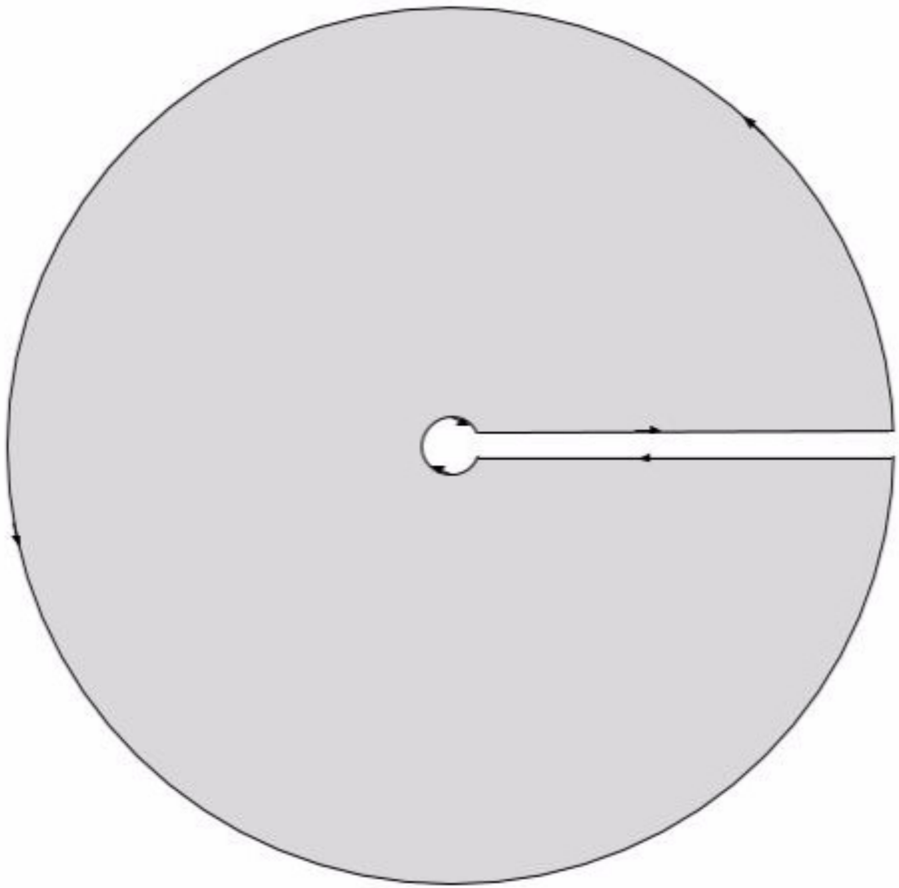

Figure: A keyhole contour.

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Take two line segments $\eta_+(t) = t + i\delta$, $\eta_-(t) = (R - t) - i\delta$ where $t \in [a, b]$ such that $\eta_+(a), \eta_-(b) \in C_\epsilon$, $\eta_+(b), \eta_-(a) \in C_R$.

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Let γ_R be the positively oriented path on the circle of radius R joining the endpoints of η_+ and η_- on that circle and similarly let γ_ϵ the path on the circle of radius ϵ which is negatively oriented and joins the endpoints of $\gamma_\pm$ on the circle of radius ϵ.

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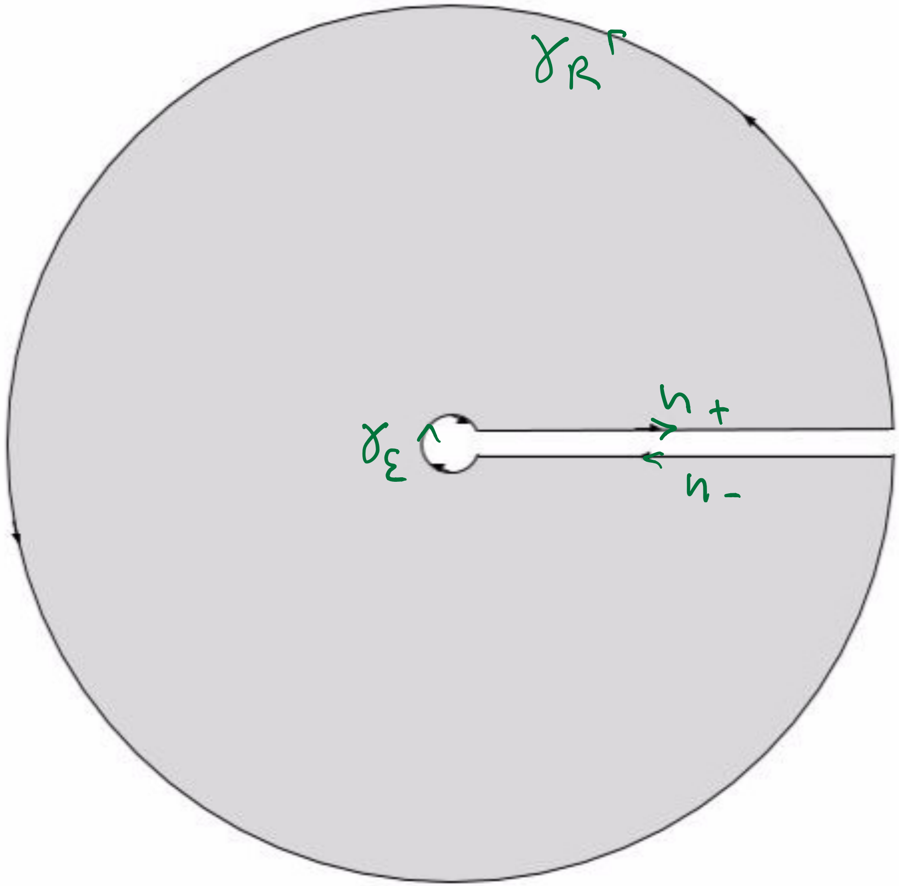
We let $\epsilon \to 0$ and $R \to \infty$.



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since

for $z = re^{i\theta} \in \eta_-$, $z^{1/2} \sim r^{1/2}e^{i\pi} = -r^{1/2}$ and η_- is traversed in the opposite direction from η_+.

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$$\int_{\eta_-} z^{1/2}/(1+z^2)dz \to \int_0^\infty \frac{x^{1/2}}{1+x^2}dx$$

since

for $z = re^{i\theta} \in \eta_-$, $z^{1/2} \sim r^{1/2}e^{i\pi} = -r^{1/2}$ and η_- is traversed in the opposite direction from η_+.

We use the residue theorem: The function $f(z)$ has simple poles at $z = \pm i$. We calculate the residues:

$$\lim_{z\to i}(z-i)z^{1/2}/(1+z^2) = \frac{1}{2}e^{-\pi i/4},$$

$$\lim_{z\to -i}(z+i)z^{1/2}/(1+z^2) = \frac{1}{2}e^{5\pi i/4}.$$

It follows that

$$\int_{\Gamma_{R,\epsilon}} f(z)dz = 2\pi i\left(\frac{1}{2}e^{-\pi i/4} + \frac{1}{2}e^{5\pi i/4}\right) = \pi\sqrt{2}.$$

Taking the limit as $R \to \infty$ and $\epsilon \to 0$ we see that $2\int_0^\infty \frac{x^{1/2}}{1+x^2}dx = \pi\sqrt{2}$, so that

$$\int_0^\infty \frac{x^{1/2}dx}{1+x^2} = \frac{\pi}{\sqrt{2}}.$$

Conformal transformations

Informally if $U, V \subseteq \mathbb{C}$, $T : U \to V$ is conformal if it preserves the angles at each point.

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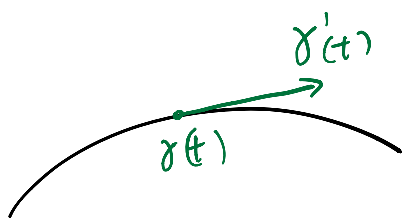
To make sense of this recall

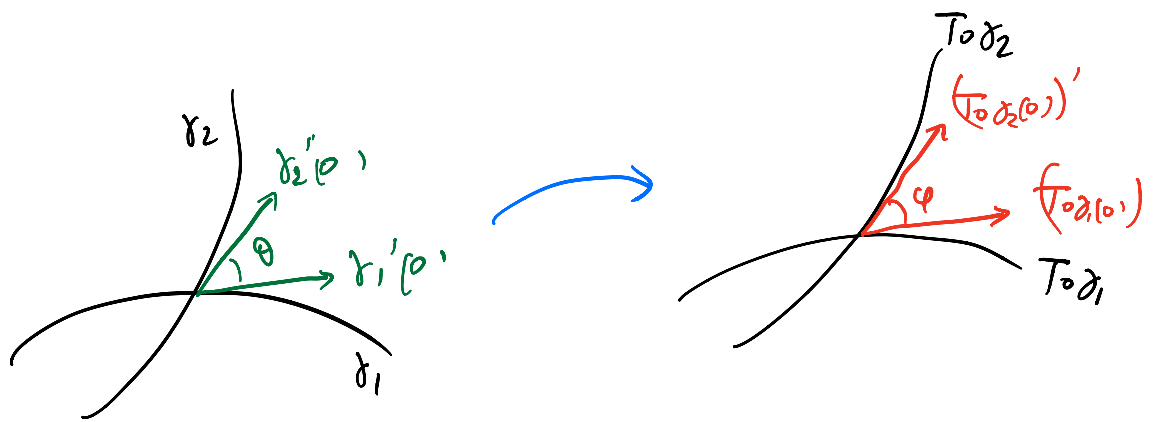
Definition

If $\gamma : [-1, 1] \to \mathbb{C}$ is a C^1 path which has $\gamma'(t) \neq 0$ for all t, then we say that the line $\{\gamma(t) + s\gamma'(t) : s \in \mathbb{R}\}$ is the *tangent line* to γ at $\gamma(t)$, and the vector $\gamma'(t)$ is a tangent vector at $\gamma(t) \in \mathbb{C}$.



Definition

Let U be an open subset of $\mathbb{C}$ and suppose that $T\colon U \to \mathbb{C}$ is continuously differentiable in the real sense (so all its partial derivatives exist and are continuous). If $\gamma_1, \gamma_2\colon [-1,1] \to U$ are two paths with $z_0 = \gamma_1(0) = \gamma_2(0)$ then $\gamma_1'(0)$ and $\gamma_2'(0)$ are two tangent vectors at z_0, and we may consider the angle between them (formally speaking this is the difference of their arguments).



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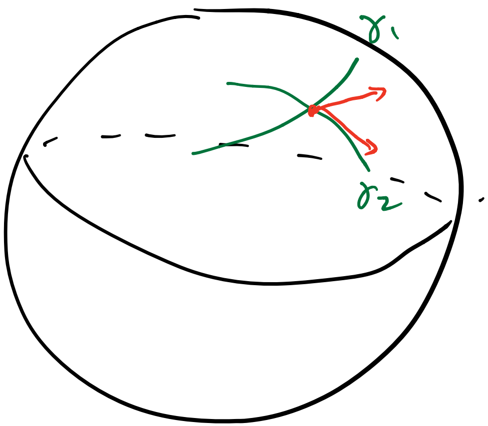
Remark

Note that we can define tangent vectors at points on subsets of $\mathbb{R}^n$ using C^1-paths (ie all component functions are C^1).
In particular, if $\mathbb{S}$ is the unit sphere in $\mathbb{R}^3$ we consider C^1 paths on $\mathbb{S}$ ie paths $\gamma\colon [a, b] \to \mathbb{R}^3$ whose image lies in $\mathbb{S}$.



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It is easy to check that the tangent vectors at a point $p \in \mathbb{S}$ all lie in the plane perpendicular to p – simply differentiate the identity $f(\gamma(t)) = 1$ where $f(x, y, z) = x^2 + y^2 + z^2$ using the chain rule to get

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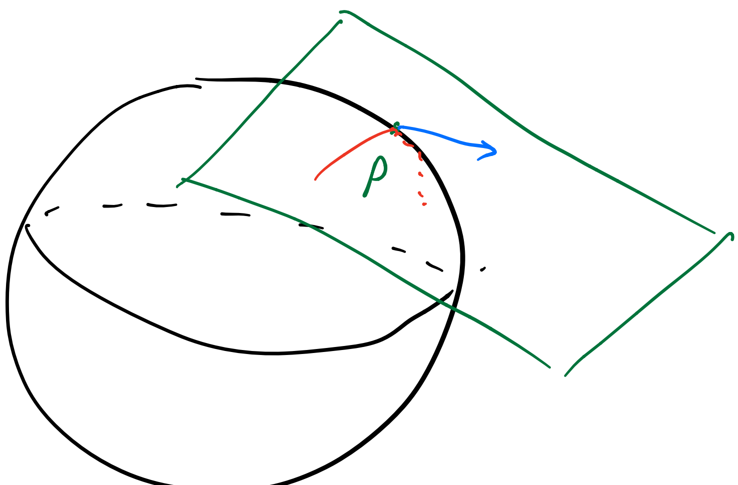

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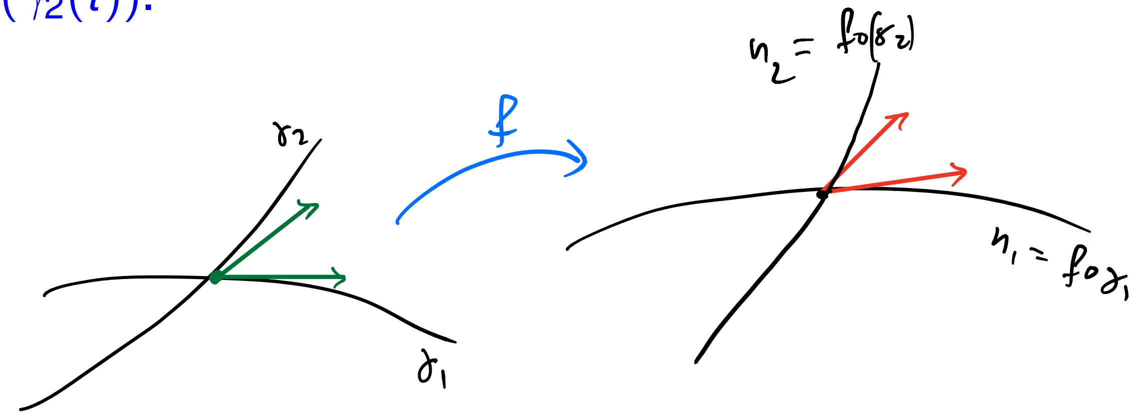
Proposition

Let $f\colon U \to \mathbb{C}$ be a holomorphic map and let $z_0 \in U$ be such that $f'(z_0) \neq 0$. Then f is conformal at z_0. In particular, if $f\colon U \to \mathbb{C}$ has nonvanishing derivative on all of U, it is conformal on all of U (and locally a biholomorphism).

Proof.

Let γ_1 and γ_2 be C^1-paths with $\gamma_1(0) = \gamma_2(0) = z_0$. Then we obtain paths η_1, η_2 through $f(z_0)$ where $\eta_1(t) = f(\gamma_1(t))$ and $\eta_2(t) = f(\gamma_2(t))$.



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We show that a version of the chain rule applies to these compositions. For $i = 1, 2$ we have

$$\eta_i'(0) = \lim_{h\to 0} \frac{f(\gamma_i(h)) - f(\gamma(0))}{h} = \lim_{h\to 0} \frac{f(\gamma_i(h)) - f(z_0)}{\gamma_i(h) - z_0} \cdot \frac{\gamma_i(h) - z_0}{h}$$

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Clearly for small h, $\gamma_i(h) \neq z_0$ as $\gamma_i'(0) \neq 0$ and $\lim_{h\to 0} \dfrac{f(\gamma_i(h)) - f(z_0)}{\gamma_i(h) - z_0} = f'(z_0)$.

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So if we set $f'(z_0) = \rho e^{i\theta}$ we have

$$\eta_i'(0) = f'(z_0)\gamma_i'(0) = \rho e^{i\theta}\gamma_i'(0), \quad i = 1, 2.$$

so if

$$\gamma_1'(0) = r_1 e^{i\phi_1}, \quad \gamma_2'(0) = r_2 e^{i\phi_2}$$

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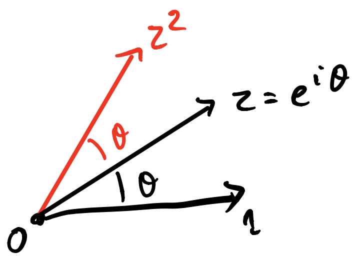
For the final part, note that if $f'(z_0) \neq 0$ then $f(z)$ is locally biholomorphic by the inverse function theorem. $\square$

Example

The function $f(z) = z^2$ has $f'(z)$ nonzero everywhere except the origin. It follows f is a conformal map from $\mathbb{C}^\times$ to itself. Note that the condition that $f'(z)$ is non-zero is necessary – if we consider the function $f(z) = z^2$ at $z = 0$, $f'(z) = 2z$ which vanishes precisely at $z = 0$, and it is easy to check that at the origin f in fact doubles the angles between tangent vectors.



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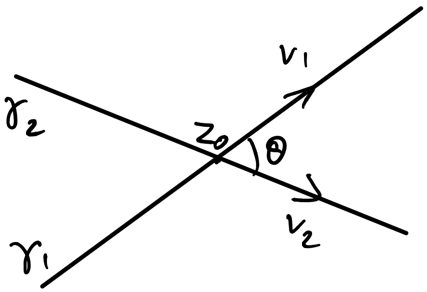

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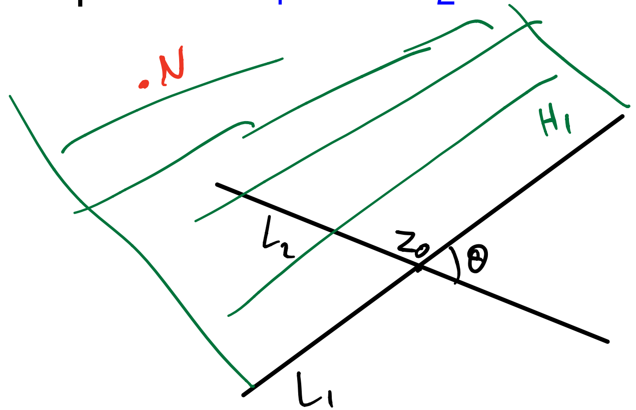
Then the lines L_1 and L_2 they describe, together with north pole of $\mathbb{S}$, N, determine planes H_1 and H_2 in $\mathbb{R}^3$.



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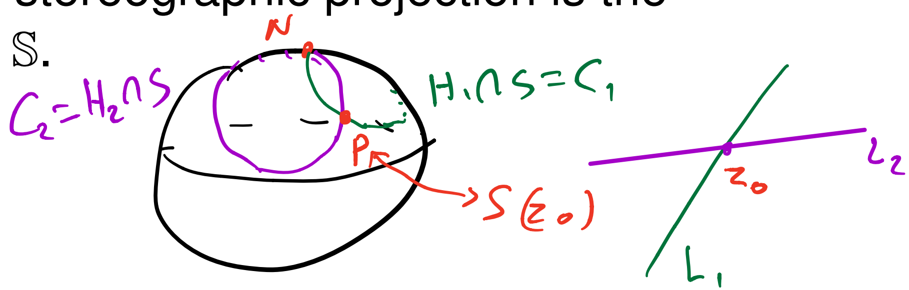
The image of L_1, L_2 under stereographic projection is the intersection of H_1, H_2 with $\mathbb{S}$.



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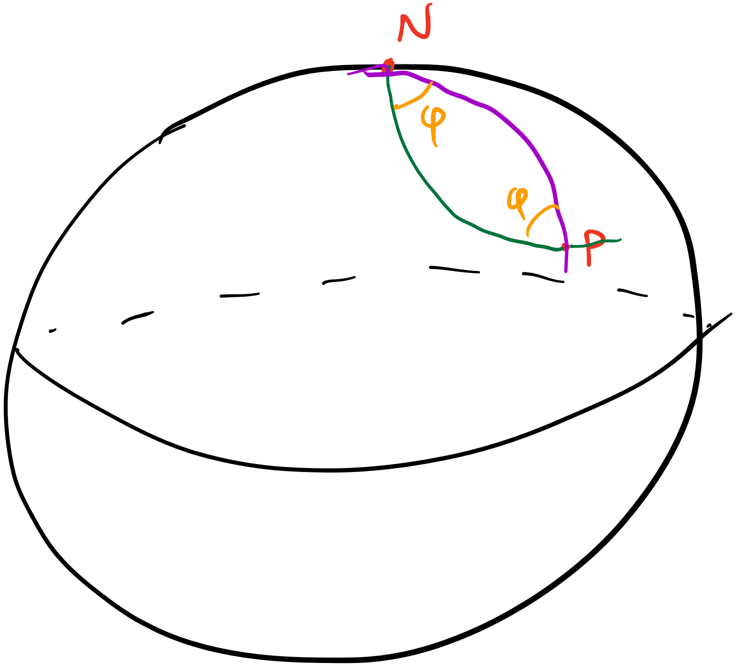
So the paths γ_1 and γ_2 get sent to two circles C_1 and C_2 passing through $P = S(z_0)$ and N.

By symmetry, C_1, C_2 meet at the same angle at N as they do at P.



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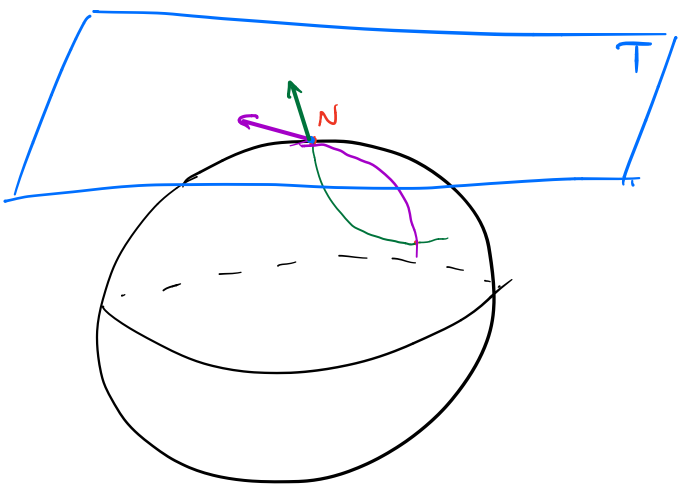
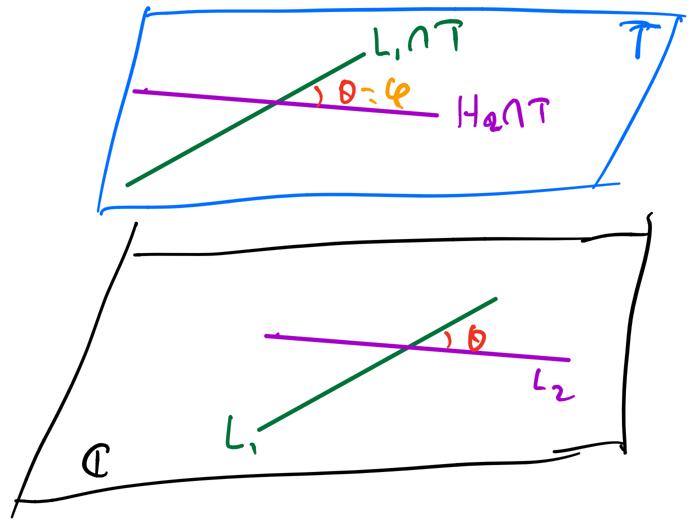
The tangent lines of C_1 and C_2 at N are just the intersections of H_1 and H_2 with the plane tangent to $\mathbb{S}$ at N.





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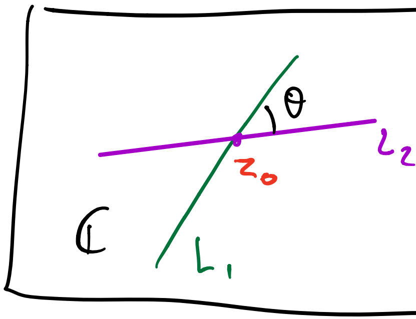
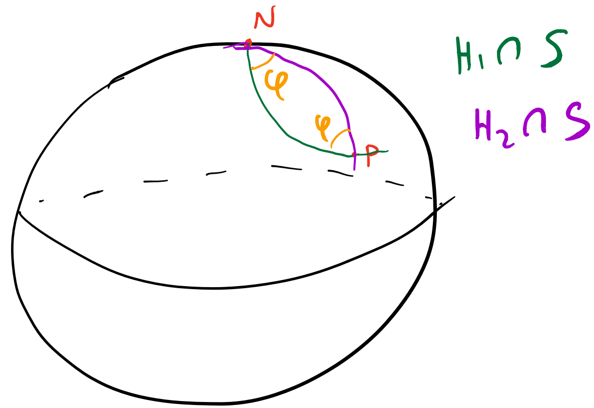
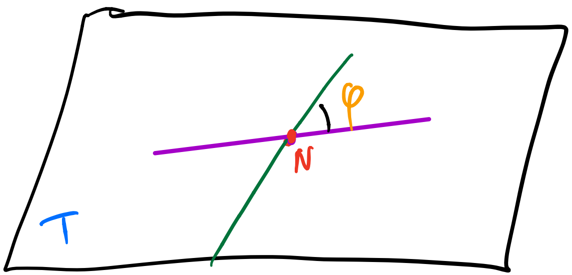
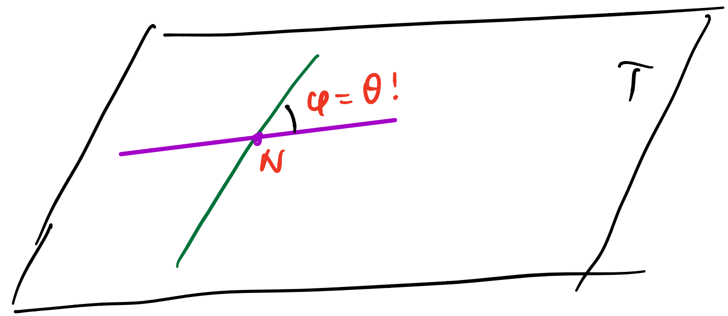
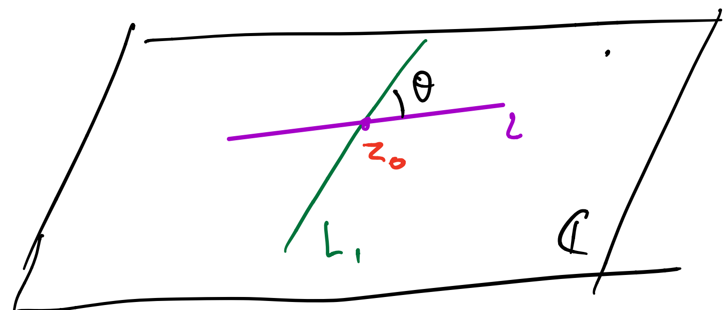
But this means the angle between them will be the same as that between the intersection of H_1 and H_2 with the complex plane, since it is parallel to the tangent plane of $\mathbb{S}$ at N. Thus the angles between C_1 and C_2 at P and L_1 and L_2 at z_0 coincide as required. $\square$











$H_2 \cap T$

$H_1 \cap T$

T is tangent to S at N

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Möbius transformations are conformal

We would like to see Möbius transformations as maps from the extended complex plane $\mathbb{C} \cup \{\infty\} = \mathbb{S}$ to itself.

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We note that if f is conformal at z_1 and g is conformal at $f(z_1)$ then $g \circ f$ is conformal at z_1. Since the stereographic projection is conformal a map $f : \mathbb{C} \to \mathbb{C}$ is conformal if and only if the corresponding map $f : \mathbb{S} \to \mathbb{S}$ is conformal.

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We claim that $1/z$ seen as a map $\mathbb{S} \to \mathbb{S}$ is conformal. Indeed $1/z : \mathbb{S} \to \mathbb{S}$ is the map $(t, u, v) \mapsto (t, -u, -v)$, which is a rotation by π about the x-axis, so clearly it is conformal.

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We claim that $z \mapsto z + a$ and $z \mapsto az$ are also conformal maps for $a \in \mathbb{C} \setminus \{0\}$.

The maps $z \mapsto z + a$, $z \mapsto az$ $(a \neq 0)$ are clearly conformal for every $z \in \mathbb{C}$, so they are conformal at every $z \in \mathbb{S} \setminus \{N\}$

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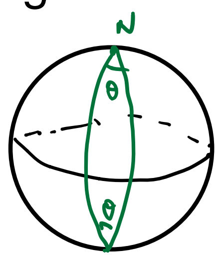
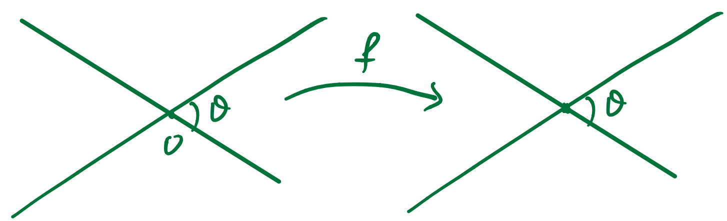
We claim that if f is $z \mapsto z + a$ or $z \mapsto az$ then f is conformal at N as well.

To see this we consider the images of great circles through N. These circles correspond to lines through 0 under S and as in the previous lemma we note that the angles of two such circles at N is equal to the angle of the lines at 0. But, since f is conformal as a map $\mathbb{C} \to \mathbb{C}$ the angles at 0 are preserved by f, so the angles at N are preserved as well.





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Proof.

We note that if $f(z) = \frac{az+b}{cz+d}$ then

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for all $z \neq -d/c$, thus f is conformal at each $z \in \mathbb{C} \backslash \{-d/c\}$.

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We claim further that a Möbius transformation is conformal seen as a map $\mathbb{S} \to \mathbb{S}$ (where $\mathbb{S}$ can be identified with $\mathbb{C} \cup \infty$). Indeed we have seen that any Möbius transformation can be written as a composition of dilations, translations and an inversion. Since all these are conformal maps $\mathbb{S} \to \mathbb{S}$ their compositions are conformal as well. So Möbius tranformations are conformal. $\square$

Proposition

If z_1, z_2, z_3 *and* w_1, w_2, w_3 *are triples of pairwise distinct complex numbers, then there is a unique Möbius transformation* f *such that* $f(z_i) = w_i$ *for each* $i = 1, 2, 3$.

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Proof. It is enough to show that, given any triple (z_1, z_2, z_3) of complex numbers, we can find a Möbius transformations which takes z_1, z_2, z_3 to $0, 1, \infty$ respectively.

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Now consider

$$f(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$$

It is easy to check that $f(z_1) = 0, f(z_2) = 1, f(z_3) = \infty$, and clearly f is a Möbius transformation as required.

If $z_1 = \infty$ then we set $f(z) = \frac{z_2 - z_3}{z - z_3}$; if $z_2 = \infty$, we take $f(z) = \frac{z - z_1}{z - z_3}$; and finally if $z_3 = \infty$ take $f(z) = \frac{z - z_1}{z_2 - z_1}$.

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But suppose $T(z) = \dfrac{az + b}{cz + d}$ is Möbius with $T(0) = 0$, $T(1) = 1$ and $T(\infty) = \infty$. Since T fixes ∞ it follows $c = 0$. Since $T(0) = 0$ it follows that $b/d = 0$ hence $b = 0$, thus $T(z) = a/d \cdot z$, and since $T(1) = 1$ it follows $a/d = 1$ and hence $T(z) = z$.

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Hence

$$hf_1g^{-1} = hf_2g^{-1} = \mathrm{id},$$

and so $f_1 = f_2$. $\square$

Examples of conformal maps

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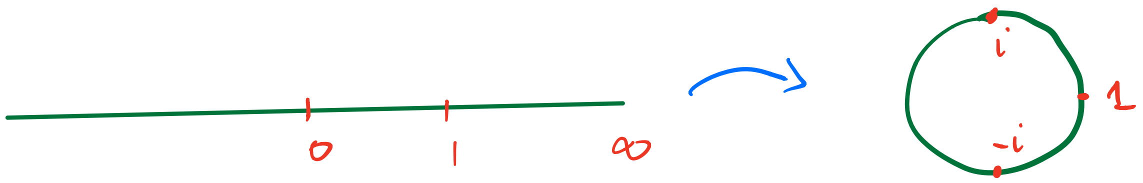
Take f the Möbius defined by $0 \mapsto -i$, $1 \mapsto 1$, $\infty \mapsto i$. Then the real axis is sent to the unit circle.



We calculate:

$$f(z) = \frac{iz+1}{z+i}$$

$f(z) = \dfrac{az+b}{cz+d}$ $\quad f(0) = \dfrac{b}{d} = -i$ $\quad f(\infty) = \dfrac{a}{c} = i$ $\quad f(1) = \dfrac{a+b}{c+d} = 1$

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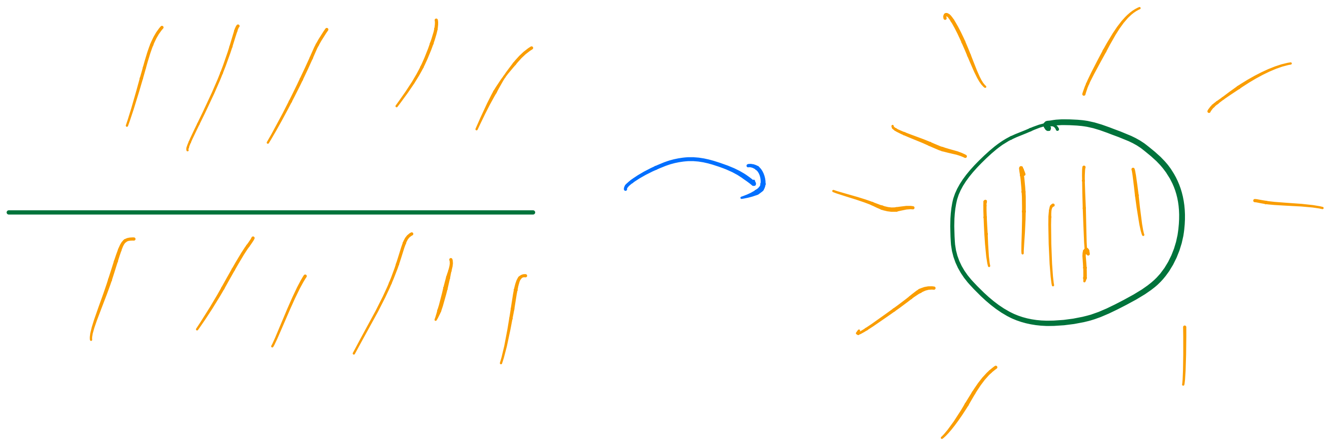

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We calculate $f(i) = 0 \in B(0, 1)$, so $f(\mathbb{H}) = B(0, 1)$.

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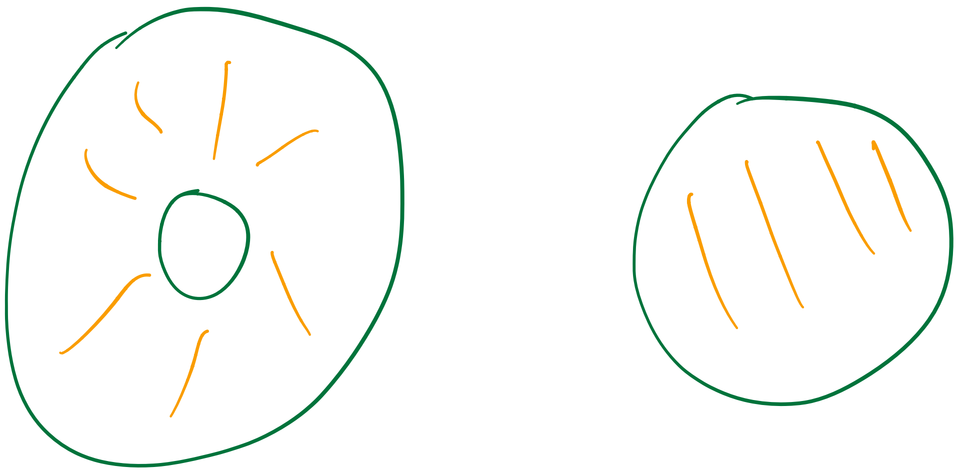
Definition

If there is a bijective conformal transformation between two domains U and V in the complex plane then we say that they are conformally equivalent.

Riemann mapping theorem

Since two conformally equivalent domains are in particular homeomorphic, one can not expect that any two domains are conformally equivalent.



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Liouville's theorem implies that there can be no bijective conformal transformation taking $\mathbb{C}$ to $B(0,1)$, so the whole complex plane is an exception.

Say D_1, D_2 are open proper simply connected subsets. How do we construct $f : D_1 \to D_2$ conformal?

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Some useful maps: Möbius transformations, the exponential function, branches of the multifunction $[z^\alpha]$ (away from the origin).

More examples of conformal maps

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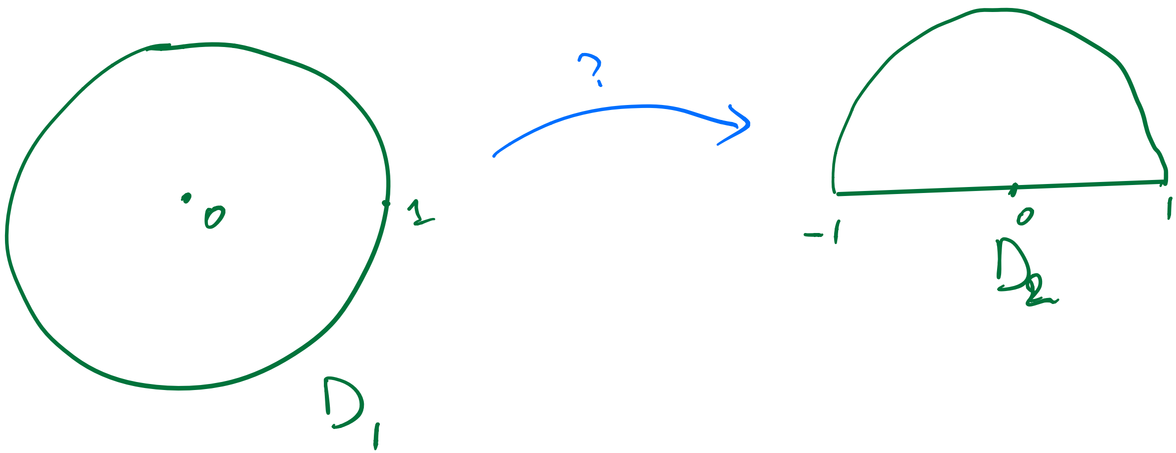
Example.

Let $D_1 = B(0,1)$ and $D_2 = \{z \in \mathbb{C} : |z| < 1, \Im(z) > 0\}$. Since these domains are both convex, they are simply-connected, so by Riemann's mapping theorem there is a conformal map sending D_2 to D_1.



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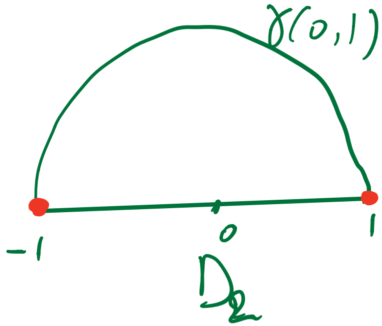

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Since f is Möbius and $f(-1) = \infty$, $f(1) = 0$ both $\gamma(0,1)$, $[-1,1]$ map to half lines from 0.

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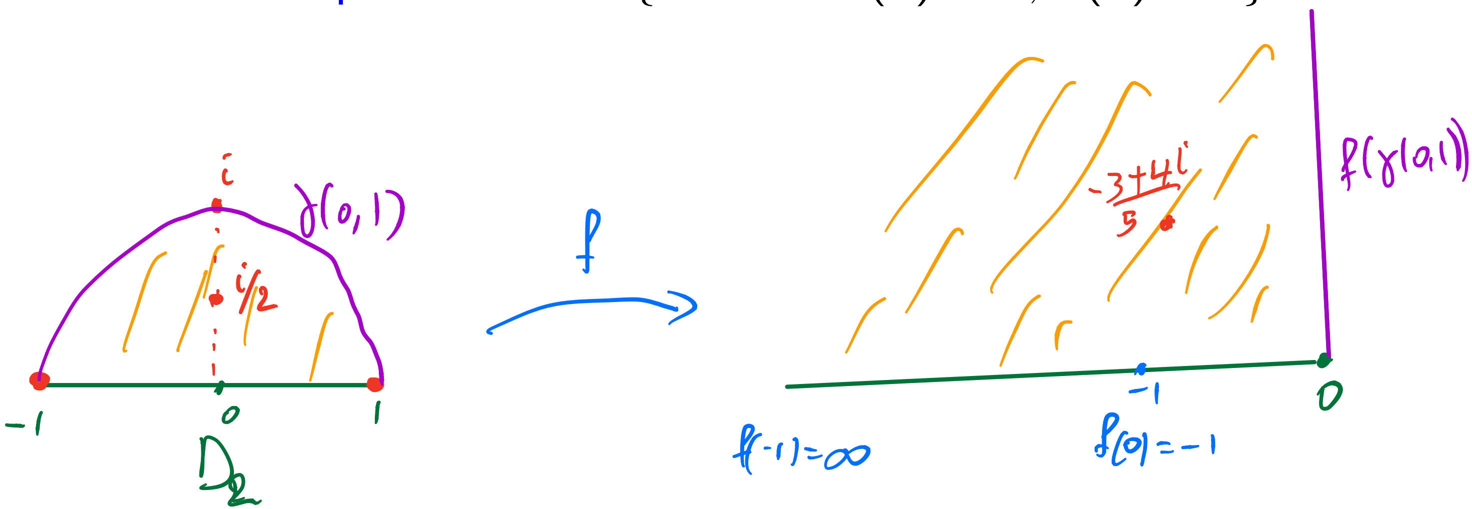

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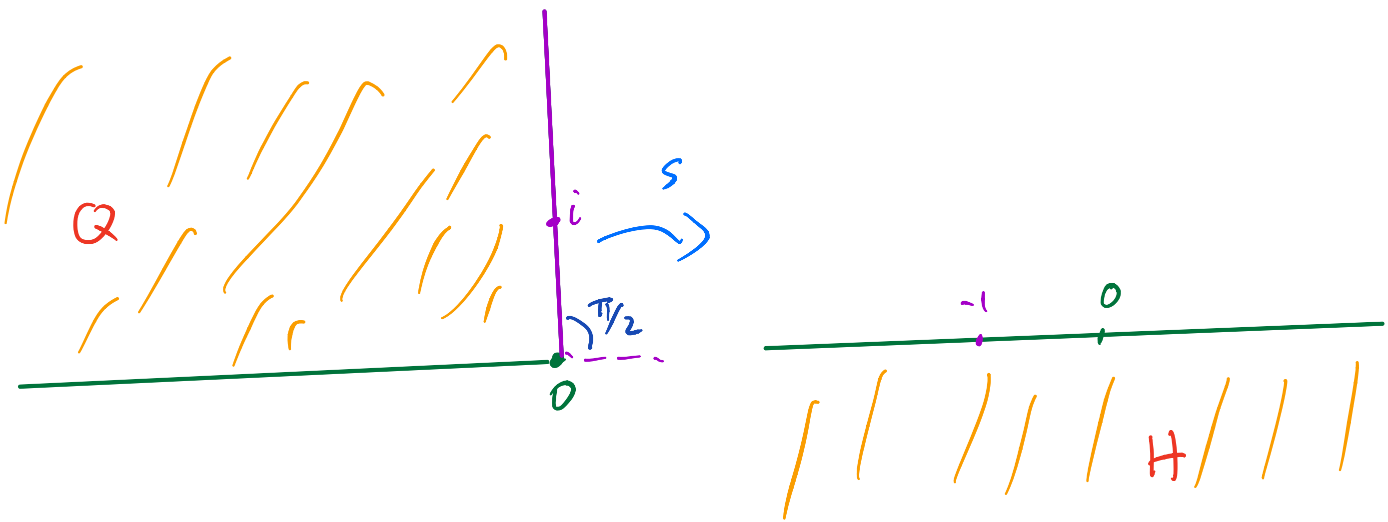

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$$F(z) = i \left(\frac{z^2 + 2iz + 1}{z^2 - 2iz + 1} \right)$$

General principles: If we have circular arcs on the boundary we may transform them to half-lines by Möbius transformations that map one of the endpoints to ∞.

Branches of fractional power maps $[z^\alpha]$ allow us to change the angle at the points of intersection of arcs of the boundary.

Möbius transformations allow us to map half planes to discs.

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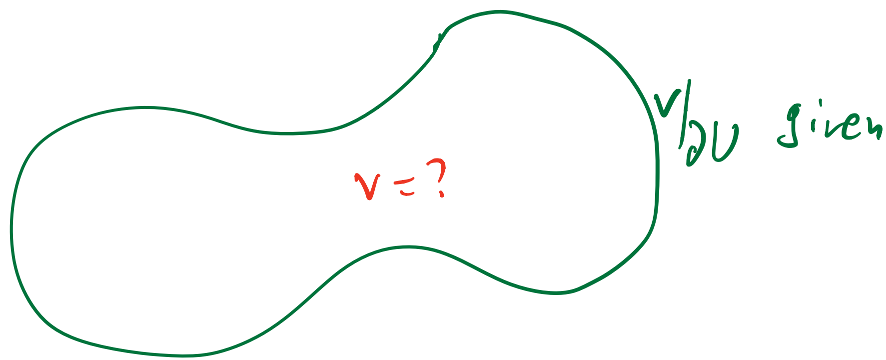

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Lemma

Suppose that $U \subset \mathbb{C}$ is a simply-connected open subset of $\mathbb{C}$ and $v \colon U \to \mathbb{R}$ is twice continuously differentiable and harmonic. Then there is a holomorphic function $f \colon U \to \mathbb{C}$ such that $\Re(f) = v$. In particular, any such function v is analytic.

Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

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ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

Recall

$$\left.\begin{aligned} & f = u + iw \\ \text{and} \quad & \partial_x u = \partial_y w \\ & \partial_x w = -\partial_y u \\ & u, w \ C^2 \end{aligned}\right\} \Rightarrow f \text{ holomorphic}$$

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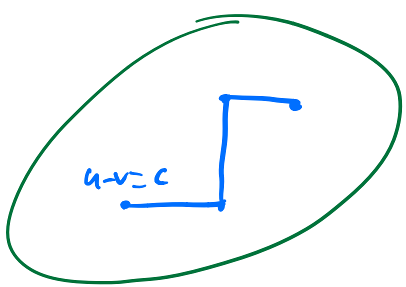
$G' = \partial_x u + i\partial_x w = -i\partial_y u + \partial_y w$ so $\partial_x u = \partial_x v$, $\partial_y u = \partial_y v$. It follows that u, v differ by a constant on each vertical and on each horizontal path.



Proof.

(sketch)Consider the function $g(z) = \partial_x v - i\partial_y v$. Then since v is twice continuously differentiable, the partial derivatives of g are continuous and

$$\partial_x^2 v = -\partial_y^2 v; \quad \partial_y \partial_x v = \partial_x \partial_y v,$$

ie g satisfies the Cauchy-Riemann equations, hence g is holomorphic.

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It follows that $u - v = c$ a constant and v is the real part of $f = G - c$.

Dirichlet problem and holomorphic maps

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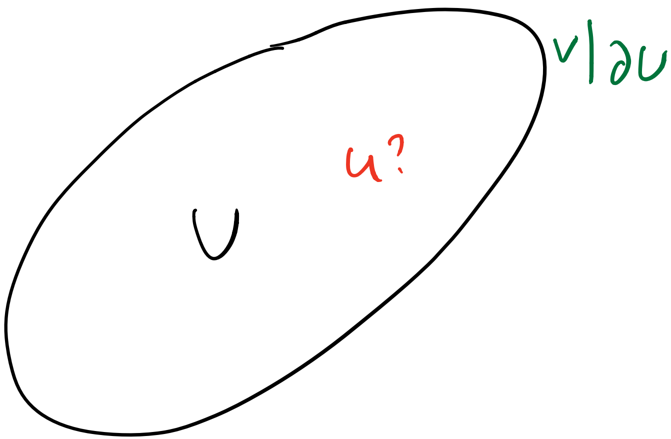
Recall the Dirichlet Problem: Given a continuous function v on ∂U for some domain U find a harmonic function u extending v to U. So u is continuous on $\overline{U}$ and equal to v on ∂U.



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We showed in the last lecture that if u is a harmonic function on a simply connected domain U then u is the real part of a holomorphic function. Conversely given a holomorphic function f we obtain a harmonic function by taking its real part.

So to solve the Dirichlet problem for a simply connected domain U for a given function g on ∂U, it suffices to find a holomorphic function f on U such that $\Re(f) = g$ on the boundary ∂U.

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Precisely we have:

Lemma

If U and V are domains and $G\colon U \to V$ is a conformal transformation, then if $u\colon V \to \mathbb{R}$ is a harmonic function on V, the composition $u \circ G$ is harmonic on U.

Proof.

To see that $u \circ G$ is harmonic we need only check this in a disk $B(z_0, r) \subseteq U$ about any point $z_0 \in U$.

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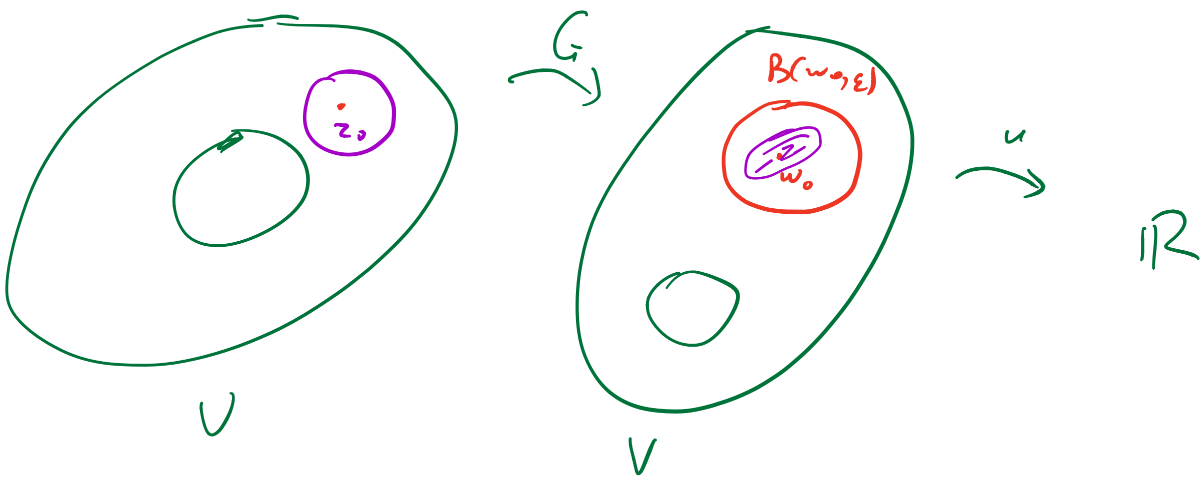

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But then on $B(z_0, \delta)$ we have $u \circ G = \Re(f \circ G)$, and by the chain rule $f \circ G$ is holomorphic, so its real part is harmonic. $\square$

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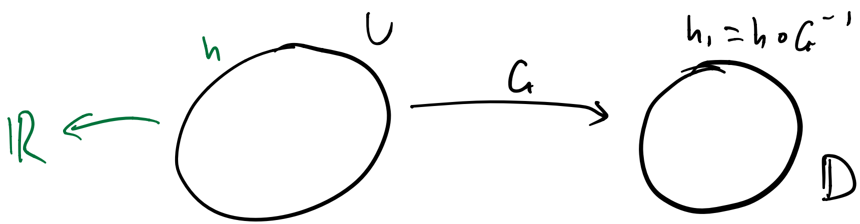

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For the solution of Dirichlet's problem one needs something slightly stronger:

Theorem

Let U, V be bounded domains in $\mathbb{C}$ and let $f : U \to V$ be a conformal map. If $\partial U, \partial V$ are piecewise C^1 Jordan curves the conformal map $f : U \to V$ can be extended to a homeomorphism $\bar{f} : \bar{U} \to \bar{V}$.

(for a proof see the book Introduction to Complex Analysis by K. Kodaira, p. 215)

Step 2: Suppose that u is a harmonic function defined on $B(0, r)$ for some $r > 1$. Then there is a holomorphic function $f\colon B(0, r) \to \mathbb{C}$ such that $u = \Re(f)$.

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By Cauchy's integral formula, if γ is a parametrization of the positively oriented unit circle, then for all $w\in B(0,1)$ we have $f(w)=\frac{1}{2\pi i}\int_\gamma f(z)/(z-w)dz$, and so

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Since the integrand uses only the values of f on the boundary circle, we have almost recovered the function u from its values on the boundary. But we need the values of f rather than u on the boundary. The next lemma gives an expression that only depends on u.

Lemma

If u is harmonic on $B(0,r)$ for $r > 1$ then for all $w \in B(0,1)$ we have

$$u(w) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta = \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\Re\left(\frac{e^{i\theta}+w}{e^{i\theta}-w}\right)d\theta.$$

Proof (*Sketch.*) Let $f(z)$ be holomorphic with $\Re(f) = u$ on $B(0,r)$. Then letting γ be a parametrization of the positively oriented unit circle we have

$$f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-w} - \frac{1}{2\pi i}\int_\gamma \frac{f(z)dz}{z-\bar{w}^{-1}}$$

where the first term is $f(w)$ by the integral formula and the second term is zero because $f(z)/(z-\bar{w}^{-1})$ is holomorphic inside all of $B(0,1)$. So

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$\boxed{\bar z = \frac{1}{z}}$

$$\frac{1}{z-w} - \frac{1}{z-\bar w^{-1}} = \frac{\cancel{z} - \frac{1}{\bar w} - \cancel{z} + w}{z(1-w\bar z)\frac{(z\bar w - 1)}{\bar w}} = \frac{1}{z}\cdot\frac{1-|w|^2}{|1-w\bar z|^2}$$

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The real part is

$$u(w) = \int_0^{2\pi} u(e^{i\theta})\frac{1-|w|^2}{|e^{i\theta}-w|^2}d\theta.$$

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It remains to show that as $z \to z_0 \in \partial\mathbb{D}$, $u(z) \to h(z_0)$ for all $z_0 \in \partial\mathbb{D}$. We refer to the book Complex Analysis by Ahlfors sec. 6, thm 23 for this.