A2: Metric Spaces and Complex Analysis

Sheet 1 (Metric Spaces, Chapters 1–3) — MT20

1. Let $\Sigma = \{0,1\}^{\mathbf{N}}$ be the set of all sequences of 0s and 1s. If $\sigma = (a_n)_{n=1}^{\infty}$, $\sigma' = (b_n)_{n=1}^{\infty} \in \Sigma$ define
$$d(\sigma, \sigma') = \frac{1}{\min\{n : a_n \neq b_n\}},$$
where the right-hand side is to be interpreted as zero when $a_n = b_n$ for all n). Show that (Σ, d) is a metric space.

2. Let $\Omega = \mathbb{R}^{\mathbb{N}}$ be the space of all real sequences. If $\mathbf{x} = (x_n)_{n=1}^{\infty}$, $\mathbf{y} = (y_n)_{n=1}^{\infty} \in \Omega$, define
$$d(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|x_n - y_n|}{1 + |x_n - y_n|}.$$
Show that this defines a metric on Ω.

3. Let $X = \mathbf{R}^n$ and suppose that d is one of d_1, d_2, d_∞. Suppose we have two balls $B(x_1, 0.9), B(x_2, 0.9)$, both contained in $B(0,1)$. Show that they intersect. Is the same true in a general metric space (X, d)?

4. Let (X, d) be a metric space, and let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. Define $\tilde{d}$ by
$$\tilde{d}((x_1, x_2), (y_1, y_2)) = \|(d(x_1, y_1), d(x_2, y_2))\|.$$
Show that $\tilde{d}$ is a metric on $X \times X$ if $\|\cdot\|$ is the ℓ^1-norm or the ℓ^∞-norm. Is this in fact true for an arbitrary norm $\|\cdot\|$?

5. Consider the 2-adic metric on the integers defined in the lecture notes. Show that the sequence $9, 99, 999, \ldots$ converges.

6. Let X be a metric space and suppose $f\colon X \to \mathbb{R}$ is continuous. Show that if $f(a) \neq 0$, then there is an $\varepsilon > 0$ such that $1/f$ is defined and is continuous on $B(a, \varepsilon)$.

7. Suppose that (X, d) is a metric space and that $X \times X$ is endowed with the product metric defined in lectures. Show that the metric d, viewed as a map from $X \times X$ to $\mathbf{R}$, is continuous.

8. Show that the map $f : \mathbf{Z} \to \mathbf{Z}$ defined by $f(n) = n^2$ is continuous in the 2-adic metric.

9. Let X, Y and Z be metric spaces and equip $Y \times Z$ with the product metric defined in lectures. If $F\colon X \to Y \times Z$ is a function and we write $F(x) = (f_1(x), f_2(x))$, show that F is continuous if and only if f_1 and f_2 are continuous.

10. Let X denote the vector space of sequences $\mathbf{x} = (x_n)_{n=1}^{\infty}$ with $x_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} x_n^2 < \infty$. Explain why $\|\mathbf{x}\|_\infty := \sup_n |x_n|$ is a well-defined norm on X. Is the metric induced by this norm equivalent to the metric induced by the ℓ^2-norm?

Mathematical Institute, University of Oxford
Ben Green: ben.green@maths.ox.ac.uk