A2: Metric Spaces and Complex Analysis

Sheet 2 (Metric spaces, Chapters 4–6) — MT20

1. Let X be a metric space. A collection $\mathcal{U}$ of open sets in X is said to be a *basis for the topology on* X if every open set in X is a union of sets from $\mathcal{U}$. Show that the collection of all open intervals whose length is 2^{-m} for some $m \geqslant 1$ is a basis for the topology on $\mathbf{R}$.

2. Let X be a metric space. Show that the closure of the union of two subsets A, B of X is the union of the closures of A and B, that is $\overline{A \cup B} = \bar{A} \cup \bar{B}$. Is $\bar{A} \cap \bar{B} = \overline{A \cap B}$?

3. Let $A \subseteq [0,1]$ be the set of real numbers which can be expressed in the form $\sum_{n=1}^{\infty} \frac{a_n}{3^n}$ where $a_n \in \{0,1\}$. Show that A is closed. [*You may assume without proof that every real number in* $[0,1]$ *has a ternary* (*base 3*) *expansion, which is unique except for recurring 2s* (*thus, for example,* $0.11 = \frac{4}{9}$, $0.222\cdots = 1$.)]

4. Let $A \subseteq \mathbf{R}$. For this question, write $i(A)$ for the interior of a set A and $c(A)$ for its closure in $\mathbf{R}$.
   (i) Find $i(A)$ and $c(A)$ when $A = (0,1) \cup (1,2]$, and when $A = \mathbf{Q} \cap (0,1)$.
   (ii) Give an example of a set A such that 7 different sets (including A itself) can be obtained by applying the $i()$ and $c()$ operations in some order.
   (iii) Show that for every A we have $icic(A) = ic(A)$ and $cici(A) = ci(A)$.
   (iv) Show that for every A at most 7 different sets (including A itself) can be obtained by applying the $i()$ and $c()$ operations in some order.

5. Let X be a metric space, and suppose that A and B are disjoint closed subsets of X. Define $\mathrm{dist}(A,B) := \inf_{a \in A, b \in B} d(a,b)$. Show that if A is a singleton then $d(A,B) > 0$, but that this is not true in general.

6. Show that, in $C[0,1]$, the functions with $f(\mathbf{Q}) \subseteq \mathbf{Q}$ are dense.

7. Let X be the set of functions in $C[-1,1]$ which are differentiable on $(-1,1)$, together with the sup norm $\|f\|_\infty = \sup_x |f(x)|$. Is X complete?

8. Consider $X = C[-1,1]$ with the metric defined by the norm $\|f\|_1 = \int_{-1}^{1} |f(t)| dt$. Is X complete?

9. Show that the space Σ considered in Sheet 1, Q1 is complete.

10. Show that $\mathbf{Z}$ is not complete with the 2-adic metric.

11. Let X be a complete metric space, and let $A_1, A_2, \ldots$ be a sequence of dense open sets in X. Show that $\bigcap_{n=1}^{\infty} A_n$ is nonempty.

Ben Green: ben.green@maths.ox.ac.uk