A2: Metric Spaces and Complex Analysis

Sheet 3 (Metric Spaces, Chapters 7–9) — MT20

1. Let $X = C[0,1]$, with the usual distance coming from the $\|\cdot\|_\infty$-norm. Is X connected?

2. Let $A \subseteq \mathbf{R}^2$ be the set of all points with at least one rational coordinate. Is A connected? What if the points with *both* coordinates rational are removed from A?

3. Show that there is no continuous injective map $f : \mathbf{R}^2 \to \mathbf{R}$. [*Hint: consider the restriction of f to $\mathbf{R}^2 \setminus \{a\}$, for a suitable point a*]

4. Let X be a metric space and $A_1, A_2, \ldots$ an infinite collection of subsets of X. For each of the following statements, give a proof or counterexample.

   (i) If $A_1, A_2, \ldots, A_k$ are sequentially compact then so is $A_1 \cup A_2 \cup \ldots \cup A_k$.

   (ii) If $A_1, A_2, \ldots, A_k$ are connected then $A_1 \cap A_2 \cap \ldots \cap A_k$ is connected.

   (iii) If $A_1, A_2, \ldots$ are sequentially compact then $\bigcup_{k \geqslant 1} A_k$ is sequentially compact.

   (iv) If $A_1, A_2, \ldots$ are connected and $A_j \cap A_{j+1} \neq \emptyset$ then $\bigcup_{k \geqslant 1} A_k$ is connected.

5. Show that $\mathbf{Z}$ with the 2-adic metric is not connected.

6. Suppose that X is connected and that $f : X \to \mathbf{R}$ is *locally constant*, meaning that every $x \in X$ lies in some open set U on which f is constant. Show that f is constant.

7. Is there a metric on $\mathbf{N}$ which makes it connected?

8. Let $(V, \|.\|)$ be a normed vector space whose unit sphere $S = \{v \in V : \|v\| = 1\}$ is sequentially compact. Show that any closed ball $B = \{v \in V : \|v\| \leqslant R\}$ is sequentially compact. Show that V is complete.

9. Let X be a subset of $\mathbf{R}^n$ such that every continuous function $f : X \to \mathbf{R}$ is bounded. Show that X is sequentially compact.

10. Let $\|\cdot\|$ be an arbitrary norm on $\mathbf{R}^n$. Show that there is some constant C such that $\|v\| \leqslant C\|v\|_1$ for all $v \in \mathbf{R}^n$. Using this, show that there is some constant $c > 0$ such that $\|v\| \geqslant c\|v\|_1$ for all $v \in \mathbf{R}^n$. [*Hint. Consider $\|\cdot\| : \mathbf{R}^n \to \mathbf{R}$ as a function on the normed space $\mathbf{R}^n$ with the $\|\cdot\|_1$-norm.*]

11. Write down an infinite compact subset of $\mathbf{Q}$ and prove that it is compact directly from the open cover definition of compactness.

12. Consider the space Ω of all sequences $\mathbf{x} = (x_n)_{n=1}^\infty$ with $x_n \in [0,1]$ for all n, together with the metric $d(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^\infty 2^{-k}|x_k - y_k|$. Show that Ω is sequentially compact. [*Hint. You might wish to use a diagonal argument.*]

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