A2: Metric Spaces and Complex Analysis

Sheet 4 (Complex Analysis – Sections 1–3 of the notes) — MT20

1. Let $f(z) = ze^z$. Write down expressions for the components $u(x, y), v(x, y)$, and check directly that u is a harmonic function on $\mathbf{R}^2$. Do the same for $f(z) = \sin z$, giving your expressions for u and v in terms of the hyperbolic functions $\sinh, \cosh$.

2. Does $f(z) = e^z$ extend to a continuous function from $\mathbf{C}_\infty$ to itself?

3. Show that there is some $z \in \mathbf{C}$ such that $\sin z = 2$.

4. If $f : \mathbf{C} \to \mathbf{C}$ is a function, define $f^\circ(z) := \overline{f(\bar{z})}$. Show that f is holomorphic if and only if f° is.

5. Let $f\colon \mathbf{C} \to \mathbf{C}$ be the function defined by $f(x + iy) = \sqrt{|x||y|}$ for all $x, y \in \mathbf{R}$. Show that f satisfies the Cauchy-Riemann equations at 0 but is not complex differentiable there.

6. Suppose that $f : \mathbf{C} \to \mathbf{C}$ is a holomorphic function. Show that if any one of the following conditions is satisfied then f is constant: (i) $\Re(f)$ is constant; (ii) $\Im(f)$ is constant; (iii) $|f|$ is constant.

7. Let $\mathbf{C}_\infty$ be the extended complex plane, and let $\mathbb{P}^1(\mathbf{C})$ be the projective line, as in lectures. Let $\iota : \mathbf{C}_\infty \to \mathbb{P}^1(\mathbf{C})$ be the identification between these two sets as described in lectures. Let $\tilde{d}$ be the unique metric on $\mathbb{P}^1(\mathbf{C})$ such that ι is an isometry (where, as usual, $\mathbf{C}_\infty$ is given the metric d). Show that $\tilde{d}(([z_1 : w_1], [z_2 : w_2]) = 2\sqrt{1 - \frac{|\langle v_1, v_2\rangle|^2}{\|v_1\|^2\|v_2\|^2}}$, where $v_1 = (z_1, w_1)$, $v_2 = (z_2, w_2) \in \mathbf{C}^2 \setminus \{0\}$ and $\langle , \rangle$ denotes the Hermitian inner product on $\mathbf{C}^2$, that is to say $\langle (z_1, w_1), (z_2, w_2)\rangle := z_1\bar{z}_2 + w_1\bar{w}_2$, and $\|(z, w)\|^2 := \langle (z, w), (z, w)\rangle = |z|^2 + |w|^2$.

8. The *special unitary group* $\mathrm{SU}(2)$ is the subgroup of $\mathrm{GL}_2(\mathbf{C})$ consisting of matrices of the form $g = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}$, where $a, b \in \mathbf{C}$ and $|a|^2 + |b|^2 = 1$. Check that this is indeed a group under matrix multiplication. Show that if $g \in \mathrm{SU}(2)$ then the Möbius transformation Ψ_g is an isometry of $\mathbf{C}_\infty$ (*you may use the result of the previous question if you like*).

9. Show that the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ has radius of convergence 1. Let $f(z)$ be the function to which it converges on the domain $D = \{z \in \mathbf{C} : |z| < 1\}$. Show that $\exp(f(z)) = \frac{1}{1-z}$.

10. Let $S : \mathbf{C}_\infty \to \mathbb{S}$ be the isometry from $\mathbf{C}_\infty$ to the unit sphere $\mathbb{S} \subseteq \mathbf{R}^3$ described in lectures. Show that S maps circlines in $\mathbf{C}_\infty$ to circles in $\mathbb{S}$, and vice versa.

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