A2: Metric Spaces and Complex Analysis

Sheet 5, sections 4-7.1 from the notes — MT20

1. Consider the multifunction $[F(z)] = (z^2 - 1)^{1/2}$. It was asserted in lectures that $\{1, -1\}$ were branch points for this multifunction. Prove this carefully, using the branches which were constructed in lectures.

   [*Hint: Suppose, for the case of* -1*, there was a continuous branch* $f(z)$ *of* $[F(z)]$ *defined on some* $B(-1, r)\backslash\{-1\}$*. Compare this to a branch as we constructed in lectures to obtain a contradiction.*]

2. *i)* Suppose that $l(z)$ is holomorphic on $\mathbb{C}\backslash(-\infty, 0]$ and satisfies $\exp l(z) = z$. Show that

   $$l(z) = L(z) + 2n\pi i$$

   for some $n \in \mathbb{Z}$ where $L(z)$ is the holomorphic branch of log defined in lectures.

   ii) Show that there is no holomorphic function $\lambda(z)$ on $\mathbb{C}\backslash\{0\}$ such that $\exp \lambda(z) = z$.

   iii) There are unique holomorphic branches of $\log z$, $\sqrt{z}$ and $\sqrt[3]{z}$ on the cut plane $\mathbb{C}\backslash$ {negative imaginary axis} such that $\log 1 = 0$; $\sqrt{1} = 1$; $\sqrt[3]{1} = 1$. For these branches determine

   $$\log(1+i), \qquad \sqrt{-1-i}, \qquad \sqrt[3]{-2}, \qquad \sqrt{1-i}.$$

   iv) Let C denote the logarithmic spiral given in polar coordinates by $r = 2e^{\theta}$. There is a unique holomorphic branch of log on $\mathbb{C}\backslash C$ such that $\log 1 = 0$. For this branch determine

   $$\log i, \qquad \log 3, \qquad \log(-1), \qquad \log 1000, \qquad \log(-1000), \qquad \log 2000.$$

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3. *Green's Theorem* states, for a region D in the plane, bounded by (an) oriented closed curve(s)[1] C in $\mathbb{R}^2$ and for real-vaued L and M with continuous partial derivatives on D, then
$$\int_C (L\,\mathrm{d}x + M\,\mathrm{d}y) = \int\int_D (M_x - L_y)\,\mathrm{d}x\,\mathrm{d}y.$$
If we assume, for a holomorphic function $f = u + iv$, that u_x, u_y, v_x, v_y are continuous, show that Cauchy's Theorem follows from Green's Theorem, that is, show that for a function f which is holomorphic on the interior D of a closed curve C, we have $\int_C f(z)dz = 0$.

   [*The terms "positively oriented" and "interior" should be interpreted as they were in multivariable calculus. We will discuss them more rigorously later in the course.*]

   In the following questions, for $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ we let $\gamma(a, r)$ denote the positively oriented circle centred at a of radius $r > 0$.

4. By making the substitution $z = re^{i\theta}$, and making clear any special cases, for each integer k determine $\int_{\gamma(0,r)} z^k\,\mathrm{d}z$ (where as usual $\gamma(0, r)$ is the path $\gamma(0, r)(t) = re^{it}$ for $t \in [0, 2\pi]$). By writing $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/2i$ rewrite the integral on the left as a path integral around $\gamma(0, 1)$ and deduce that
$$\int_0^{2\pi} \sin^{2n}\theta\,\mathrm{d}\theta = \frac{2\pi}{4^n}\binom{2n}{n}.$$

5. Use the estimation lemma to show that, if $\gamma\colon [0, 1] \to \mathbb{C}$ is a piece-wise C^1 closed path, then the winding number $I(\gamma, z)$ is constant on the connected components of $\mathbb{C}\backslash\gamma^*$.

   [*Hint: Since it is integer-valued, it suffices to show that $I(\gamma, z)$ is a continuous function on $\mathbb{C}\backslash\gamma^*$.*]

6. Suppose that $f\colon U \to \mathbb{C}$ is holomorphic on $U\backslash\{p\}$ for some $p \in U$, and that f is bounded near p (thus there are constants $r, K \in \mathbb{R}_{>0}$ such that $|f(z)| < K$ for all $z \in B(p, r)$). Show that if T is any triangle whose interior is entirely contained in U then Cauchy's theorem for a triangle still holds, that is
$$\int_T f(z)dz = 0.$$
[*Hint: Use Cauchy's theorem for a triangle to shrink the size of the triangle T.*]

[1] Note that the boundary of a region in the plane, for example "with holes", may be a disjoint union of closed curves.

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7. Use Cauchy's Integral Formula and the holomorphic function $f(z) = z^n(z-a)^{-1}$, where $a \in \mathbb{R}$ with $a > 1$, to calculate the integral:

$$\int_0^{2\pi} \frac{\cos(n\theta)}{1 - 2a\cos(\theta) + a^2} d\theta.$$

8. Suppose that D is a domain bounded by a contour C, which we assume can be parameterized by a function $\gamma_1 \colon [0,1] \to \mathbb{C}$ (that is, $C = \gamma_1^*$). Let $z_0 \in D$ and let $r > 0$ be small enough so that $\bar{B}(z_0, r) \subset D$. The region $D \backslash \bar{B}(z_0, r)$ is thus bounded by $C \cup \partial B(z_0, r)$. Use the result of question 3 to show that if f is holomorphic on $D \backslash \{z_0\}$ then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz,$$

where $\gamma_2(t) = z_0 + re^{it}$, $(0 \le t \le 2\pi)$.

Use this and question 4 to calculate

$$\int_0^{2\pi} \frac{dt}{a^2\cos^2(t) + b^2\sin^2(t)}.$$

9. (Optional)In this question you should compute winding numbers "by eye".

   1. Compute the value of the winding number of the path γ in the connected components of $\mathbb{C} \backslash \gamma^*$ in the diagram below.
   2. Suppose that $a, b \in \mathbb{C}$ and $|a| < r < |b|$. Compute using the integral formula for the winding number, the integral

$$\int_{\gamma(0,r)} \frac{dz}{(z-a)(z-b)}.$$

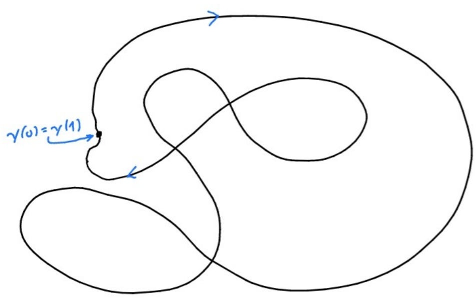

Figure 1: The path γ for question 9.

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