A2: Metric Spaces and Complex Analysis

Sheet 7, sections 10-11.3 — MT20

1. Suppose that f is a holomorphic function defined on an open set U of the complex plane containing $\bar{B}(0,1)$. Let $S^1 = \{z \in \mathbb{C} : |z| = 1\} = \partial B(0,1)$. Show that if $f(S^1)$ is an ellipse and f restricted on S^1 is injective, then f is injective on $\bar{B}(0,1)$. [*Hint: Apply the argument principle to* $f(z) - w_0$ *for* $w_0 \in \mathbb{C}$.]

2. Prove, for $a > 0$, that
$$\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4 + a^4} = \frac{\pi}{a^3\sqrt{2}}.$$

3. Show that
$$\int_{-\infty}^{\infty} \frac{\cos \pi x}{2x - 1}\,\mathrm{d}x = -\frac{\pi}{2}.$$

4. By considering the integral
$$\int_{\Gamma_n} \frac{\pi \mathrm{d}w}{w^2 \sin \pi w}$$
where Γ_n is the square in $\mathbb{C}$ with vertices $\pm(n + 1/2)(1 \pm i)$ show that
$$\frac{\pi^2}{12} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots$$
(You may assume that there exists C such that $|\csc \pi w| \leqslant C$ on Γ_n for all n and all w.)

5. Write down a definition of a branch of $\log(z + i)$ which is holomorphic in the cut-plane
$$\mathbb{C}\backslash\{z : \operatorname{Re} z = 0,\ \operatorname{Im} z \leqslant -1\}.$$
By integrating $\log(z + i)/(z^2 + 1)$ around a suitable closed path, evaluate
$$\int_{-\infty}^{\infty} \frac{\log(x + i)}{x^2 + 1}\,\mathrm{d}x$$
and, by taking real parts, show that
$$\int_{-\infty}^{\infty} \frac{\log(x^2 + 1)}{x^2 + 1}\,\mathrm{d}x = 2\pi \log 2.$$

6. Show that
$$\int_{0}^{\infty} \frac{\sin px \sin qx}{x^2}\,\mathrm{d}x = \frac{\pi \min(p,q)}{2},$$
where $p, q > 0$.

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7. Let $a \in \mathbb{C}$ with $-1 < \operatorname{Re} a < 1$. By considering a rectangular contour with corners at R, $R + i\pi$, $-R + i\pi$, $-R$, show that

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{\cosh x} \, \mathrm{d}x = \pi \sec\left(\frac{\pi a}{2}\right)$$

and hence evaluate, for real n,

$$\int_{-\infty}^{\infty} \frac{\cos nx}{\cosh x} \, \mathrm{d}x.$$

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