4.3 Well-posedness of problems

We call a problem (usually consisting of a PDE & some type of data)

well-posed if

- existence
- uniqueness
- continuous dependance on data

for solutions of our problem.

In situations where we consider a PDE and where we prescribe e.g.

$$w \text{ on } \Gamma \leftarrow \text{some curve}$$

$$\left(\& \text{ potentially } \frac{\partial w}{\partial n} \text{ on } \Gamma \right)$$

and consider a bounded region Ω of xy-plane

we say that the solution depends continuously on data if

$\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if

$w, \tilde{w}$ solve (PDE)

and $w = f$, $\tilde{w} = \tilde{f}$ on Γ

$$\left(\frac{\partial w}{\partial n} = g, \ \frac{\partial \tilde{w}}{\partial n} = \tilde{g} \text{ on } \Gamma \right)$$

then

$$\sup_{\Gamma} |f - \tilde{f}| < \delta \Rightarrow \sup_{\Omega} |w - \tilde{w}| < \varepsilon$$

(and $\sup_{\Gamma} |g - \tilde{g}| < \delta$

Wellposedness for 3 model problems:

Wave equation:

$w_{tt} = w_{xx}$ (1 spatial D)

$w_{tt} = \Delta w = w_{xx} + w_{yy}$ (2 sp. D)

Initial value problem

(IVP) (PDE) & prescribed

$w(\cdot, t=0)$, $w_t(\cdot, t=0)$

or initial and boundary value problem

$w(\cdot, t=0)$, $w_t(\cdot, t=0)$ on bounded Ω < interval / domain $\subset \mathbb{R}^2$

& $w((x,y), t)$ $x,y \in \partial\Omega$

resp.

$w(0,t)$, $w(L,t)$ if $\Omega = [0,L]$

(IVP) and (IBVP) are wellposed.

1 spatial dim:

(IVP) $w_{tt} = w_{xx}$

$w(x,0) = f(x)$, $w_t(x,0) = g(x)$

D'Alembert's formula gives

$$w(x,t) = \frac{1}{2}[f(x+t) + f(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

= unique sol. of problem

& depends continuously on f, g.

For (IBVP)

$$w_{tt} = w_{xx}$$

$$w(t=0) = f \qquad w_t(t=0) = g$$

on $[0, L]$

$$w(0,t) = 0 = w(L,t)$$

can use separ. of variables & Fourier series to see that

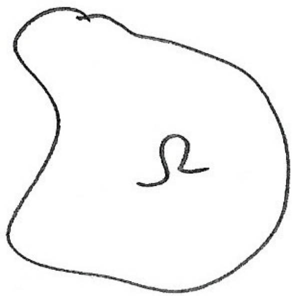
$$w(x,t) = \sum \sin\left(\frac{n\pi x}{L}\right) \cdot \left(a_n \cdot \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right)$$

unique sol.
& this depends cont. on f, g

Laplace's equation:

$$\Delta w = w_{xx} + w_{yy} = 0 \quad \text{or} \quad = f(x,y) \text{ (Poisson's eq)}$$

is well-posed when we consider (PDE) on a bounded domain $\Omega \subset \mathbb{R}^2$ and prescribe $w|_{\partial\Omega} = f$..



We'll prove
- uniqueness
- cont. dependence on data

using Maximum principle ($\to$ 4.4).

Proof of existence $\to$ part C $\left(E(w) = \frac{1}{2}\int w_x^2 + w_y^2\right)$

(IVP) for Laplace's eq. is ill-posed:

Consider $\Delta w = 0$ in $\mathbb{R} \times [0, Y]$

with $w(x,0) = 0$, $w_y(x,0) = \frac{1}{n}\sin(nx) = g_n$

$\frac{1}{n} = \sup |g_n| \to 0$

So if problem was wellposed

$\to$ would need corresp. sol.

$\to 0 \;\hat{=}$ solution for data $= 0$.

Can check that solution for g_n

are $w_n(x,y) = \frac{1}{n^2}\sin(nx)\sinh(ny)$

$$\sup_{\mathbb{R}\times[0,Y]} |w_n| \xrightarrow[n\to\infty]{} \infty.$$

Similarly (Fourier series) can show that (IBVP) is ill-posed for elliptic problems.

Heat equation

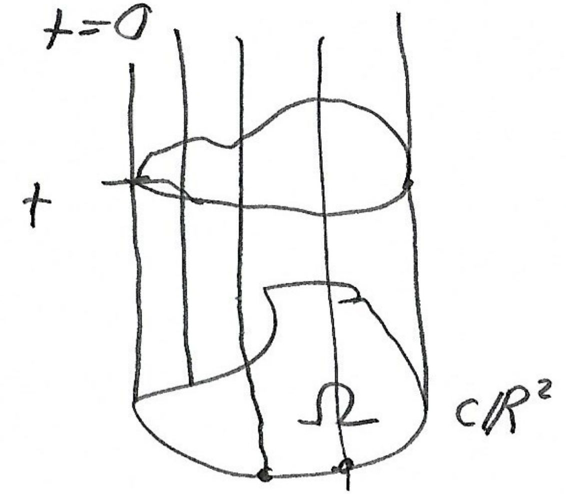
$w_t = w_{xx}$ (1 spatial D)

$w_t = \Delta w$ (2 - - - -)

(IBVP) for heat eq. is to prescribe

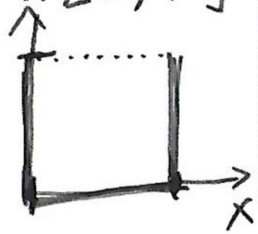
$w = g$ at $t = 0$

$w(\cdot, t) = \dots$ on $\partial\Omega$



1D case:

$$w_t = w_{xx} \quad \text{on } [0,L] \times [0,T]$$



prescribe

$$w(x, t=0) = g(x)$$

$$w(0,t) = \dots \quad w(L,t) = \dots$$

This will be wellposed

- existence (beyond this course)
- uniqueness
- cont. dep. on data

] Maximum principle

Ill-posed: • (BVP) (can't additionally prescribe $w(x,T)$)

• in reverse time direction

Note (IVP) is also wellposed

(if we ask for some reasonable behaviour at ∞ to guarantee uniqueness)

Analogue "result"/properties hold more generally for LINEAR 2nd order PDEs.

	IVP & IBVP	BVP
elliptic	✗	✓
hyperbolic	✓	✗
parabolic (forward in time)	✓	✗