4.4. The maximum principle

Elliptic case:

Consider

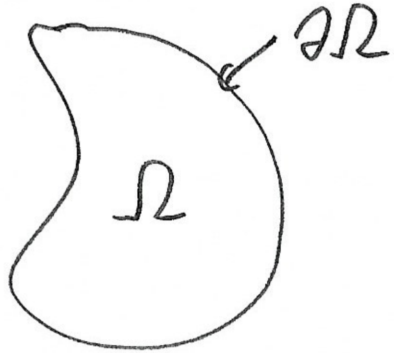
$$\Delta w = w_{xx} + w_{yy} = \text{-----}$$

and we know that r.h-s has a sign then we'll see that the maximum resp. the minimum of w on a bounded domain is achieved on the boundary.

Note: We want to work on a domain

$$\Omega = \text{bounded open set} \subset \mathbb{R}^2$$

and with functions $w : \overline{\Omega} \to \mathbb{R}$ which are continuous and 2-times differ. in Ω.



As $\overline{\Omega}$ is compact w is continuous we know that w achieves min and max on $\overline{\Omega}$.

Also $\partial\Omega$ is closed & bounded so compact so also $w|_{\partial\Omega}$ will achieve min & max and in general

$$\max_{\partial\Omega} w \leq \max_{\overline{\Omega}} w$$

Theorem 4.1 (Maximum principle, elliptic case)

Suppose Ω, w are as above and suppose that

$$\Delta w \geq 0 \quad \text{in } \Omega$$

(ie. $\forall (x,y) \in \Omega \quad w_{xx}(x,y) + w_{yy}(x,y) \geq 0$)

Then: w achieves its maximum over $\bar\Omega$ on $\partial\Omega$ ie.

$$\max_{\partial\Omega} w = \max_{\bar\Omega} w.$$

Proof:

Step 1: show

Claim: If $\Delta w > 0$ in Ω then $\max_{\partial\Omega} w = \max_{\bar\Omega} w$.

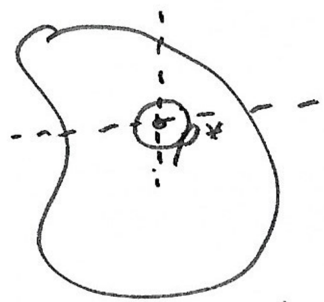
Proof of claim:

As $\bar\Omega$ compact, w is continuous
$\exists\, p^* \in \bar\Omega$ s.t. $w(p^*) = \max_{\bar\Omega} w$

To show: $p^* \in \partial\Omega$

Assume instead $p^* = (x^*, y^*) \in \Omega$

So $\nabla w(p^*) = 0$



and Hessian of w is negative semidefinite

or just note $y \mapsto w(x^*, y)$ has max

$x \mapsto w(x, y^*)$ ----

so $w_x(p^*) = 0 \quad w_{xx}(p^*) \leq 0$

$w_y(p^*) = 0 \quad w_{yy}(p^*) \leq 0$

So $\Delta w(p^*) \leq 0 \quad \lightning \quad \Delta w > 0$ in Ω.

$\blacksquare$

Step 2:

Let w be s.t. $\Delta w \geq 0$ in Ω.

Let $\underbrace{w_\varepsilon(x,y)}_{\geq w} = w + \varepsilon \cdot x^2, \ \varepsilon > 0$

Then $\Delta w_\varepsilon = \Delta w + 2 \cdot \varepsilon > 0$

so by claim

$$\max_{\bar\Omega} w \leq \max_{\bar\Omega} w_\varepsilon = \max_{\partial\Omega} w_\varepsilon \leq \max_{\partial\Omega} w + \varepsilon \cdot \max_{\partial\Omega} x^2$$

$\bar\Omega$ bounded $\to C := \max_{\partial\Omega} x^2 < \infty$

so conclude that $\forall \varepsilon > 0$

$$\max_{\bar\Omega} w \leq \max_{\partial\Omega} w + C \cdot \varepsilon$$

Letting $\varepsilon \to 0$ get

$$\max_{\bar\Omega} w \leq \max_{\partial\Omega} w$$

as reverse "$\geq$" trivially true $\to \blacksquare$

~~Rem~~

Remark:

- Thm 4.1 = "weak MP"

  "Strong MP": Under same assumpt. we <u>cannot</u> get a interior max unless $w = \text{const.}$

- Thm 4.1 also holds for functions w s.t.

$$\Delta w + a(x,y)\, w_x + b(x,y)\, w_y \geq 0$$

bounded on $\overline{\Omega}$

⌈ proof can be adjusted but need "smarter choice" of aux. function $w_\varepsilon = w + \varepsilon$ - - - -

$\Delta \ldots > 0$ much larger than $\ldots x \ \ldots y$ ⌋

Application:

If w is a solution

$$\Delta w = -e^{w^{27}} \cdot \sin w \quad \text{in } \Omega$$

and $w \geq 0$ on $\partial\Omega$

then $w \geq 0$ in Ω.

Proof: $\Delta(-w) = e^{\cdots} \geq 0$

so $\max\limits_{\overline{\Omega}}(-w) = \max\limits_{\partial\Omega}(-w) \leq 0$

so $-w \leq 0$ i.e. $w \geq 0$.

Important application

Cor. 4.2:

Consider the Dirichlet Problem for Poisson equation

$$(\text{BVP})\begin{cases} \Delta w = f(x,y) & \text{in } \Omega \\ w = g & \text{on } \partial\Omega \end{cases}$$

Then: (i) If a solution exists then it's unique.

(ii) We have continuous dependance on data, i.e. g.

Proof: (i) Let w_1, w_2 be two sol. (to same g)

Then $w = w_1 - w_2$ solves

$$\begin{cases} \Delta w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

So MP applied for w, $\Delta w = 0 \geq 0$

gives $\max_{\bar\Omega} w = \max_{\partial\Omega} w = 0$

so $w \leq 0$ on Ω.

MP for $-w$ gives as $\Delta(-w) = 0 \geq 0$

$$\max_{\bar\Omega}(-w) = \max_{\partial\Omega}(-w) = 0$$

$-w \leq 0$ so $w \geq 0$

So $\boxed{w = 0}$ i.e. $w_1 = w_2$

$\boxed{\checkmark}$ uniq.

(ii) Let $\varepsilon > 0$ and let $\delta = \varepsilon \ldots > 0$

Then suppose that w_1, w_2 are solutions of (BVP) to data g_1, g_2 where

$$\sup_{\partial\Omega} |g_1 - g_2| \leq \delta.$$

So as $w = w_1 - w_2$ solves $\Delta w = 0$ can apply MP to both w and $-w$ to get

$$w(x,y) \leq \max_{\Omega} w \overset{MP}{\leq} \max_{\partial\Omega} w$$

$$= \max_{\partial\Omega} g_1 - g_2 \leq \delta$$

$$-w(x,y) \leq \max_{\partial\Omega} g_2 - g_1 \leq \delta$$

so

$|w(x,y)| \leq \delta = \varepsilon$ and thus

$$\sup_{\overline{\Omega}} |w_1(x,y) - w_2(x,y)| \leq \varepsilon$$

$\varepsilon > 0$ was arbitrary $\boxed{///}$ cont. dep.