A1 Differential Equations I: MT 2020/21 Sheet 4

Second order semi-linear PDEs

Questions 4.1-4.4 correspond to material covered in the videos on sections 4.1-4.3 while questions 4.5 and 4.6 correspond to material covered in the videos on section 4.4.

4.1 Show that the equation

$$yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$$

is hyperbolic everywhere except on the line $y = x$. Find the characteristic variables, reduce the equation to canonical form, and show that the general solution is

$$u = \frac{1}{y-x} f(y^2 - x^2) + g(y - x).$$

4.2 Consider the partial differential equation

$$\mathrm{e}^{2y} u_{xx} + u_y = u_{yy}.$$

Write down the differential equation satisfied by its characteristic curves and show that $\phi = x + \mathrm{e}^y$ and $\psi = x - \mathrm{e}^y$ are characteristic variables for the partial differential equation.

Reduce the equation to canonical form and find the solution of the equation for which $u = x$ and $u_y = 1$ on the line $y = 0$, $0 \leq x \leq 1$.

Sketch the characteristic curves $x + e^y = 1$, $x + e^y = 2$, $x - e^y = -1$, $x - e^y = 0$.

In what region of the x, y-plane is your solution uniquely determined by the initial data? Show this region on your diagram.

4.3 Determine the type of the PDE

$$y^2 u_{xx} + x^2 u_{yy} = 0, x > 0, y > 0$$

and transform it into normal form.

4.4 Recall from Prelims that the solution of the initial-value problem

$$c^2 u_{xx} = u_{tt}$$

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad -\infty < x < \infty,$$

where f and g are prescribed functions is given by d'Alembert's formula. Use the formula to show that if $|f(x)| \leq \delta$ and $|g(x)| \leq \delta$ for $-\infty < x < \infty$ then

$$|u(x,t)| \leq (1+T)\delta \text{ for } -\infty < x < \infty, \quad 0 \leq t \leq T. \tag{1}$$

Formulate a definition of what it means for the solution u of this initial-value problem to depend continuously on the on the data f and g on any strip $\{(x,t) : -\infty < x < \infty, \quad 0 \leq t \leq T\}$. Use (1) above to show that your definition is satisfied.

4.5 (a) Let D be the region bounded by the lines $t = 0$, $t = \tau > 0$ and $x = 0$ and $x = a > 0$. Suppose further that the twice continuously differentiable function $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \tag{2}$$

in D, where f is continuous and $f \leq 0$, in D. Prove that u attains its maximum value on $x = 0$, $x = a$ or $t = 0$.

(b) (i) Hence show that, if it exists, the solution of (2) is unique if we take as boundary data

$$\begin{aligned} u(0,t) &= g(t) \quad 0 < t < \tau, \\ u(a,t) &= h(t) \quad 0 < t < \tau, \\ u(x,0) &= k(x) \quad 0 < x < a, \end{aligned}$$

where g, h and k are all continuously differentiable.

(ii) Show also that there is continuous dependence on the initial data.

(iii) Consider now the problem where the PDE and all the boundary conditions are satisfied for x between 0 and a and all positive t. Deduce that the solution of this problem is unique, if it exists.

(iv) Find all the non-negative solutions of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = -u^2,$$

with boundary data

$$\begin{aligned} u(0,t) &= 0 \quad 0 < t < \tau, \\ u(a,t) &= 0 \quad 0 < t < \tau, \\ u(x,0) &= 0 \quad 0 < x < a, \end{aligned}$$

(c) Now use the result in part (a) to show that the solution of (2) is also unique if we take as boundary data:

$$\begin{aligned} u(0,t) &= g(t) \quad 0 < t < \tau, \\ \frac{\partial u}{\partial x} + u &= h(t) \quad \text{when } x = a,\ 0 < t < \tau, \\ u(0,x) &= k(x) \quad 0 < x < a, \end{aligned}$$

where g, h and k are all continuously differentiable. [*Hint: if* $\phi = u_1 - u_2$, *where* u_1 *and* u_2 *are two solutions, show that* ϕ *can achieve neither a positive maximum nor a negative minimum on* $x = a$.]

4.6 Suppose that $u(x,t) > 0$ solves

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right) + u$$

in a region D bounded by the lines $t = 0$ and $t = \tau$ and two non-intersecting smooth curves C_1 and C_2. Prove that if a solution exists then $u(x,t)$ attains its minimum value on $t = 0$ or on one of the curves C_1 and C_2.

[Note: This shows that the maximum/minimum principal can apply to non-linear problems.]