1. Ordinary differential equations & Picard's Theorem

1.1. Introduction and Warning examples:

Consider

$$G(x, y(x), y'(x), ..., y^{(n)}(x)) = 0$$

for an unknown function $y(x)$ (of one variable $x \in I$, $x \in \mathbb{R}$)

When possible, rewrite

$$\text{(ODE)} \quad y^{(n)}(x) = F(x, y(x), y'(x), ..., y^{(n-1)}(x))$$

⌈n = order of ODE = ordinary differential equation⌋

Q: • ∃? solution

• what is the solution? / what properties does the solution have?

Initial value problems

$$\text{(IVP)} \begin{cases} \text{(ODE)} \\ + \\ y(a), y'(a), ..., y^{(n-1)}(a) \end{cases}$$

Q:
- $\exists$? solution
- uniqueness?
- continuous dependence on initial data?

of (IVP).

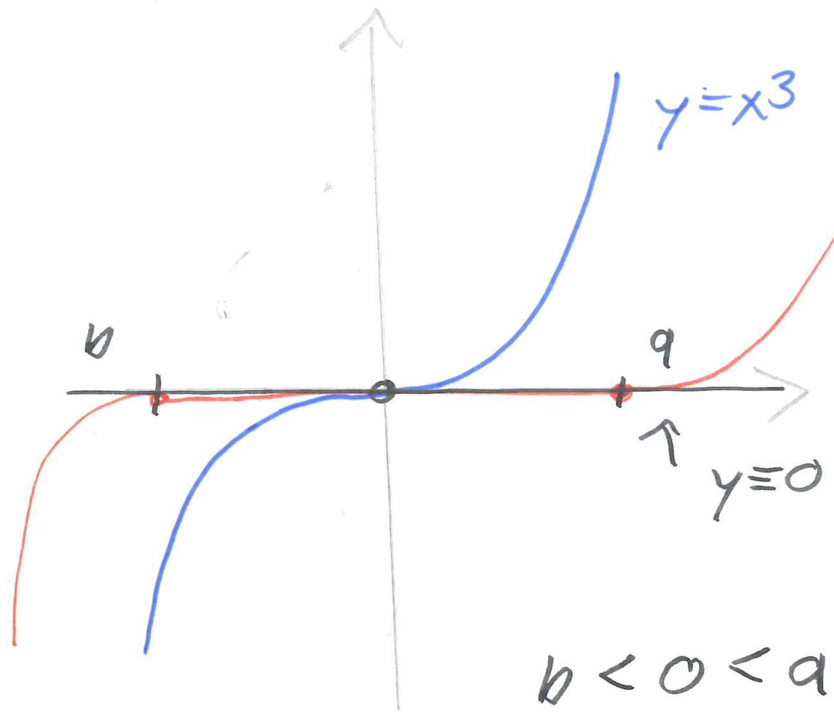
⚠ Solutions of (IVP) can be non-unique.

Ex1:
$$\begin{cases} y'(x) = 3 \cdot y^{2/3}(x) \\ y(0) = 0 \end{cases}$$

- $y(x) = 0 \quad \forall x \in \mathbb{R}.$
- Sep. var $\rightarrow \frac{1}{3}\int \frac{dy}{y^{2/3}} = \int dx$

$$y(x)^{1/3} = x + C, \; C = 0$$

So $y(x) = x^3$.



$b < 0 < a$

$$y(x) = \begin{cases} (x-b)^3 & x < b \\ 0 & b \le x \le a \\ (x-a)^3 & x > a \end{cases}$$

So (IVP) has ∞ many solutions.

⚠ Solutions of (IVP)s might not exist for all x.

Ex. 2: $\begin{cases} y'(x) = y^2(x) \\ y(0) = 1 \end{cases}$

$\underset{\text{of var.}}{\overset{\text{sep.}}{\rightsquigarrow}}$ $y(x) = \dfrac{1}{1-x}$

is only defined for

$x \in (-\infty, 1)$.

For now:

Consider

$$(IVP) \begin{cases} y'(x) = f(x, y(x)) \\ y(a) = b \end{cases}$$

where

$$f : R \longrightarrow \mathbb{R}$$

where $R = [a-h, a+h] \times [b-k, b+k]$

$h, k > 0$.

Want: Conditions on $f : R \to \mathbb{R}$ which guarantee
- $\exists$
- uniqueness

of a solution of (IVP).

1^{st} condition:

$$\boxed{f : R \to \mathbb{R} \text{ continuous}}$$

$\rightsquigarrow$ f is bounded on R,

say $|f(x,y)| \leq M \quad \forall (x,y) \in R$.

<u>Important idea</u>:

(IVP) equivalent to an integral equation:

<u>If</u> $y(x)$ solves $\underset{\text{(IVP)}}{} \left\{ \begin{array}{l} y'(x) = f(x, y(x)) \\ y(a) = b \end{array} \right.$

then

$$y(x) - y(a) = \int_a^x f(t, y(t))\,dt$$

ie. $y(x)$ satisfies

$$\text{(IE)} \quad y(x) = b + \int_a^x f(t, y(t))\,dt.$$

Conversely if y is continuous and satisfies (IE) then

$$\text{(FTC)} \quad y'(x) = f(x, y(x))$$

Additionally $y(a) = b + \int_a^a \ldots = b$.

Hence $\text{(IVP)} \iff \text{(IE)}$.

Our second condition on

$$f : R \to \mathbb{R}$$

is "Lipschitz condition" in y-var.:

<u>Def</u>: $f : R \to \mathbb{R}$ satisfies a Lip-condition on

$$R = [a-h, a+h] \times [b-k, b+k]$$

if $\exists L > 0$ s.t.

$$|f(x,y) - f(x,\tilde{y})| \leq L \cdot |y - \tilde{y}|$$

$\forall \; x \in [a-h, a+h]$

$\forall \; y, \tilde{y} \in [b-k, b+k]$.

<u>Back to Ex 2</u>:

$$\begin{cases} y'(x) = y^2(x) \\ y(0) = 1 \end{cases}$$

$$R = [-h, h] \times [1-k, 1+k]$$

$f(x,y) = y^2$ polynomial, continuous

and $|f(x,y)| = |y|^2 \leq (1+k)^2 =: M$

$$|f(x,y) - f(x,\tilde{y})| = |(y+\tilde{y}) \cdot (y - \tilde{y})|$$

$$\leq \underbrace{2(1+k)}_{=:L} \cdot |y - \tilde{y}|$$

$\forall (x,y), (x,\tilde{y}) \in R$

Lip-cond holds.

1.2 Picard's method of successive approximation:

Idea: To get a sol. of (IE)

- $y_0(x) = b$
- $y_{n+1}(x) = b + \int_a^x f(t, y_n(t))\,dt$

Hope: $y_n \xrightarrow[n\to\infty]{} y_\infty$

and y_∞ is a sol. of (IE).

Consider:

$e_0(x) = b = y_0(x)$

$e_{n+1}(x) = y_{n+1}(x) - y_n(x)$, $n \geq 0$

Note:

$$y_n(x) = \sum_{j=0}^{n} e_j(x)$$

Hope: Control $e_{n+1}(x)$ in terms the $e_n(x)$.

Note:

$$|e_{n+1}(x)| \overset{\Delta}{\leq} \left| \int_a^x |f(t, y_n(t)) - f(t, y_{n-1}(t))|\,dt \right|$$

Want to bound

$$|f(t, y_n(t)) - f(t, y_{n-1}(t))|$$

by

$$|e_n(t)| = |y_n(t) - y_{n-1}(t)|$$

Useful remark:

If $f: R \to \mathbb{R}$ is differentiable in y-direction and $\exists K$ s.t.

$$|\partial_y f(x,y)| \leq K \quad \forall (x,y) \in R$$

then Lip.-cond holds for $L = K$

since (MVT) gives:

$$|f(x,y) - f(x,\tilde{y})| \underset{\substack{\uparrow \\ \exists w \text{ between } y,\, \tilde{y}}}{=} |\partial_y f(x,w) \cdot (y - \tilde{y})|$$

$$\leq K \cdot |y - \tilde{y}|$$

⚠ f satisfying Lip cond

$\Longrightarrow$ $\partial_y f$ exists.

Ex: $f(x,y) = |y|$ $\quad (L = 1)$.