Last time:

(IVP) $\begin{cases} y'(x) = f(x, y(x)) \\ y(a) = b \end{cases}$

$f: R \to \mathbb{R}$, $R = [a-h, a+h] \times [b-k, b+k]$

Q: What conditions on f guarantee existence & uniqueness of solution of (IVP)?

Recall (IVP) equivalent to

(IE) $y(x) = b + \int_a^x f(t, y(t))\,dt$

This video:

Statement of Picard's Theorem

&

Proof via successive approximation

$y_0(x) = b$

$y_{n+1}(x) = b + \int_a^x f(t, y_n(t))\,dt.$

1.3 Picard's Theorem:

Thm 1.1 (Picard's Theorem)

Let: $f : R \to \mathbb{R}$ s.t. ($R = [a-h, a+h] \times [b-k, b+k]$)

(Pi) (a) f is continuous on R

with $|f(x,y)| \le M$

$\forall (x,y) \in R$

(b) $M \cdot h \le k$

(Pii) f satisfies a Lip. condition

(ie. $\exists L > 0$ s.t.

$|f(x,y) - f(x,\tilde{y})| \le L \cdot |y - \tilde{y}|$

$\forall (x,y), (x,\tilde{y}) \in R$

Then (IVP) **has** a **unique solution**

on $[a-h, a+h]$.

Ex. 2: $\begin{cases} y'(x) = y^2(x) \\ y(0) = 1 \end{cases}$

$R = [-h, h] \times [1-k, 1+k]$

$f(x,y) = y^2$, $M = (1+k)^2$

Lip cond ✓ $L = 2(1+k)$

$\to$ Picard guarantees $\exists!$ sol

on an interval $[-h, h]$,

h small enough s.t. e.g.

choosing $k = 1$ $\to$ $h \le \frac{k}{(1+k)^2} = \frac{1}{4}$ ..

Proof of Picard's Thm via successive approximation:

Let $y_0(x) = b$

$$y_{n+1}(x) = b + \int_a^x f(t, y_n(t))\,dt.$$

Claim 1: All y_n are well defined, continuous and

$$|y_n(x) - b| \leq k \quad \forall x \in I,$$

$I = [a-h, a+h]$.

Proof:

$n = 0$: $y_0(x) = b$ ✓

$n \rightsquigarrow n+1$:

As y_n satisfies Claim 1

- $(t, y_n(t)) \in R \quad \forall t \in I$
- y_n continuous and f continuous
- $t \mapsto f(t, y_n(t))$ well def. and continuous on I

So y_{n+1} is well defined and continuous (indeed different. by FTC)

and

$$|y_{n+1}(x) - b| \leq \left| \int_a^x \underbrace{\overbrace{|f(t, y_n(t))|}^{\in R}}_{\leq M} dt \right|$$

$$\leq M \cdot |x - a| \leq M \cdot h \leq k$$

☑ Claim 1.

Consider now

$$e_0(x) = b$$

$$e_{n+1}(x) = y_{n+1}(x) - y_n(x).$$

Claim 2:

$$|e_n(x)| \leq \frac{L^{n-1} \cdot M \cdot |x-a|^n}{n!}$$

$n \geq 1$

Proof:

n=1: $e_1(x) = \cancel{b} + \int_a^x f(t, \overset{\overset{b}{=}}{y_0(t)})\,dt$

$\cancel{-b}$

so $|e_1(x)| \leq \left| \int_a^x \underbrace{|f(t,b)|}_{\leq M} dt \right|$

$\leq M \cdot |x-a|$ ✓

$n \rightsquigarrow n+1$:

$$|e_{n+1}(x)| \overset{\Delta}{\leq} \left| \int_a^x \underbrace{|f(t,y_n(t)) - f(t,y_{n-1}(t))|}_{\leq L \cdot |y_n(t) - y_{n-1}(t)| \\ = L \cdot |e_n(t)| \\ \leq \frac{L^n \cdot M \cdot |t-a|^n}{n!}} dt \right|$$

$$\leq \frac{L^n M \cdot |x-a|^{n+1}}{(n+1)!}$$

$\boxed{\text{VI}}$ Claim 2.

Note:

$$y_n(x) = \sum_{j=0}^n e_j(x)$$

Claim 3:

y_n converge uniformly on I as $n \to \infty$ to a limit y_∞ which

- is continuous
- solves (IE).

Proof:

As $|e_n(x)| \leq \frac{L^{n-1} M}{n!} h^n =: M_n$

on $I = [a-h, a+h]$

and as $\sum M_n$ converges

Weierstrass M-test implies

$$y_n(x) = \sum_{j=0}^{n} e_j(x) \xrightarrow{n \to \infty} y_\infty(x)$$

uniformly

Note:

y_∞ is continuous since it's the uniform limit of cont. fn. y_n

Also:

$$
\begin{aligned}
y_\infty(x) &= \lim_{n \to \infty} y_{n+1}(x) \\
&= \lim_{n \to \infty} b + \int_a^x f(t, y_n(t))\,dt \\
&\overset{(*)}{=} b + \int_a^x \lim f(t, y_n(t))\,dt \\
&= b + \int_a^x f(t, y_\infty(t))\,dt
\end{aligned}
$$

So y_∞ satisfies (IE).

Note $(*)$ is ok because

$$f(t, y_n(t)) \longrightarrow f(t, y_\infty(t))$$

uniformly on I

since

$$\sup_{t\in I} \underbrace{|f(t, y_n(t)) - f(t, y_\infty(t))|} \underset{\text{Lip}}{\leq} L \cdot |y_n(t) - y_\infty(t)|$$

$$\leq L \cdot \sup |y_n(t) - y_\infty(t)|$$

$$\xrightarrow[\text{unif. conv } y_n \to y_\infty]{} 0$$

$\boxed{///}$ claim 3

$\boxed{///}$ existence.

<u>Claim 4:</u> Solution y of (IVP) is unique (among all solutions of (IVP) satisfying

$$|y(x) - b| \leq k \quad \forall x \in I,$$

i.e. which have graph in the rectangle.)

<u>Proof of claim 4:</u>

Let $y, \tilde{y}$ be two sol. of (IVP)

with $|y(x) - b| \leq k$, $|\tilde{y}(x) - b| \leq k$ on I.

Let $e(x) = y(x) - \tilde{y}(x)$.

As $y, \tilde{y}$ satisfy (IE) we have

$$e(x) = \int_a^x f(t, y(t))\,dt - \int_a^x f(t, \tilde{y}(t))\,dt$$

$$|e(x)| \leq \left| \int_a^x \underbrace{|f(t, y(t)) - f(t, \tilde{y}(t))|}_{\leq L \cdot |y(t) - \tilde{y}(t)|} dt \right|$$

$$\leq L \cdot \left| \int_a^x |e(t)|\,dt \right|$$

As $e : I \to \mathbb{R}$ continuous

so bounded. say $|e(x)| \leq C$,

get $|e(x)| \leq L \cdot C \cdot |x-a|$

$\rightsquigarrow$ - -

n-steps

$$|e(x)| \leq \frac{L^n \cdot C\, |x-a|^n}{n!} \xrightarrow[n \to \infty]{} 0 \quad \forall x \in I$$

so $e(x) = 0$

so $y = \tilde{y}$

$\boxed{///}$ claim 4.

Remark: As $M \cdot h \leq k$

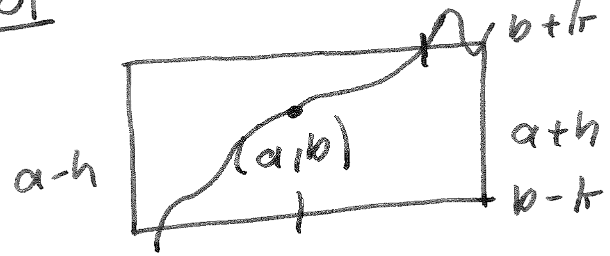
we get that every y

satisfying (IE)

has graph$(y) \subset R$

i.e. $|y(x)-b| \leq k \quad \forall x \in I$
$= [a-h, a+h]$

Proof:



Suppose y violates above property

then $\exists\, x^* \in (a-h, a+h)$ s.t.

s.t.

$|y(x^*)-b| = k$

$|y(x)-b| < k \quad \forall x$ s.t.
$|a-x| < |a-x^*|$

In particular
$(x, y(x)) \in R$ for all x
between a and x^*

so as y satisfies (IE) we get

$$y(x^*)-b = \int_a^{x^*} f(\underbrace{t, y(t)}_{\in R})\, dt$$

so
$$|y(x^*)-b| \leq \left| \int_a^{x^*} M\, dt \right|$$
$$\leq M \cdot \underbrace{|x^*-a|}_{<h}$$
$$< M \cdot h \leq k$$

$\to$ ↯ ////
remark.