Last time:

Picard's Thm:

Let $f: R = [a-h, a+h] \times [b-k, b+k] \to \mathbb{R}$

be s.t.

(Pi) f is continuous, $|f(x,y)| \le M$ on R

with $Mh \le k$

(Pii) $\exists L$ s.t.

$$|f(x,y) - f(x,\tilde{y})| \le L \cdot |y - \tilde{y}|$$

$$\forall x \in [a-h, a+h]$$

$$\forall y, \tilde{y} \in [b-k, b+k]$$

Then $\exists!$ solution of (IVP) $\begin{cases} y'(x) = f(x, y(x)) \\ y(a) = b \end{cases}$

on $[a-h, a+h]$.

Today:

- Example
- When can we expect solution of (IVP) to exist $\forall x \in \mathbb{R}$ resp $\forall x$ where f is defined?

Ex 3: $\begin{cases} y'(x) = x^2 y^{1/5}(x) \\ y(0) = b \end{cases}$

$f(x,y) = x^2 \cdot y^{1/5}$

Case 1: $b = 0$

Then: Lip-cond. is violated on every $[-h,h] \times [-k,k]$, $h,k > 0$.

Suppose not, i.e.

$\exists\, h,k,L > 0$ s.t.

$$|x^2 \cdot y^{1/5} - x^2 \tilde{y}^{1/5}| \leq L \cdot |y - \tilde{y}|$$

$\forall x \in [-h,h]$

$\forall y, \tilde{y} \in [-k,k]$.

Set $\tilde{y} = 0$, $x = h$ get $\forall y \in [-k,k]$

$$h^2 \cdot |y|^{1/5} \leq L \cdot |y|$$

so $\underbrace{|y|^{-4/5}}_{\substack{\to\infty \\ y \to 0}} \leq \dfrac{L}{h^2}$ fixed

So Picard's thm does not apply.

Case 2: $b \neq 0$

Say $b > 0$ ($b < 0$ similar)

Take $0 < k < b$ so that on

$R = [-h,h] \times [b-k, b+k]$ $\quad y \geq b-k > 0$

so $|\partial_y f(x,y)| = \frac{1}{5} x^2 |y|^{-4/5}$

$$\leq \frac{1}{5} h^2 (b-k)^{-4/5} =: L$$

so by MVT $\to$ Lip cond ok for this L.

Continuity ✓ and

$$|f(x,y)| \leq h^2 (b+k)^{1/5}$$

so (Pi) ok provided

$$h^3 \leq \frac{k}{(b+k)^{1/5}}. \quad ✓$$

(of IIVP)

$\to$ Picard gives $\exists!$ sol. on $[-h,h]$.

1.4. Global existence of solutions:

<u>Q</u>: Given $f : [c,d] \times \mathbb{R} \to \mathbb{R}$

(or $f : \mathbb{R}^2 \to \mathbb{R}$)

which is continuous and

$a \in [c,d]$, $b \in \mathbb{R}$

When can we expect that solution of (IVP) exists

$\forall x \in [c,d]$ (resp $\forall x \in \mathbb{R}$)?

⌜Ex 2 $\quad f(x,y) = y^2 \quad f : \mathbb{R}^2 \to \mathbb{R}$⌟

Need: Global Lip. cond.

<u>Claim:</u>

Let $f : [c,d] \times \mathbb{R} \to \mathbb{R}$ be continuous and s.t.

(P iii) $\exists L > 0$ s.t.

$\forall x \in [c,d]$, $\forall y, \tilde{y} \in \mathbb{R}$

we have

$$|f(x,y) - f(x,\tilde{y})| \leq L \cdot |y - \tilde{y}|.$$

Then $\exists!$ sol. of (IVP) on $[c,d]$.

<u>Adjustments of Proof of Picard needed:</u>

Won't have M s.t. $|f(x,y)| \leq M$ $\forall (x,y) \in [c,d] \times \mathbb{R}$

but don't need it

because:

(1) Don't need $|y_n(x) - b| \leq K$

$\to$ Don't need $Mh \leq K$

Still get as in Claim 1: y_n well def. cont.

(2) Bound on $e_n(x)$ still remains true if we choose

$$M = \sup_{x\in[c,d]} |f(x,\overset{y_0}{\cancel{b}})|$$

$\rightarrow$ Claim 2 still ok

Rest of proof still applies, for $x \in [c,d]$. $\rightarrow \blacksquare$

<u>Remark</u>: If $f : \mathbb{R}^2 \to \mathbb{R}$ and (Piii) is satisfied on $[a-h, a+h]$ $\forall h$ (with a const. allowed to depend on h!)

Claim $\leadsto$ $\exists!$ sol. on $[a-h, a+h]$

$\leadsto$ $\exists!$ sol on $\mathbb{R}$.

<u>Ex. 4</u>:
$$\begin{cases} y'(x) = \dfrac{x}{1+y^2(x)} \\ y(0) = b \end{cases}$$

$f(x,y) = \dfrac{x}{1+y^2} : \mathbb{R}^2 \to \mathbb{R}$ continuous

and for any $h>0$

$$|\partial_y f(x,y)| = \frac{|x|}{(1+y^2)^2}\overbrace{2|y|}^{\le 1+y^2}$$

$$\le \frac{h}{1+y^2} \le h$$

for $|x| \le h$, $y \in \mathbb{R}$.

MVT gives (Piii) holds on $[-h,h]\times\mathbb{R}$ with $L = h$

$\rightarrow$ $\exists!$ sol. on $[-h,h]$

As $h>0$ arbitrary hence $\exists!$ sol on $\mathbb{R}$.