1.5 Gronwall's inequality & continuous dependence on initial data

Recall: If $f: R \to \mathbb{R}$ satisfies (P_i) & (P_{ii}) on $R = [a-h, a+h] \times [b-k, b+k]$

then $\exists!$ solution y of

$$(IVP) \begin{cases} y'(x) = f(x, y(x)) \\ y(a) = b \end{cases}$$

For $\tilde{b}$ close to b

$\to$ still have that $(P_i), (P_{ii})$ are ok on $\tilde{R} = [a-\tilde{h}, a+\tilde{h}] \times [\tilde{b}-\tilde{k}, \tilde{b}+\tilde{k}]$

$\to \exists!\ \tilde{y}$ of

$$(\widetilde{IVP}) \begin{cases} \tilde{y}(x)' = f(x, \tilde{y}(x)) \\ \tilde{y}(a) = \tilde{b}. \end{cases}$$

<u>Q</u>: Does $\tilde{b}$ is close to b ensure that $\tilde{y}$ is close y?

<u>Note</u>: Can measure distance of two functions $y, \tilde{y} : [c, d] \to \mathbb{R}$ by

$$d(y, \tilde{y}) := \|y - \tilde{y}\|_{sup} = \sup_{x \in [c,d]} |y(x) - \tilde{y}(x)|$$

<u>Q</u>: Do we have that $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. if $|\tilde{b} - b| \leq \delta$ then $\|y - \tilde{y}\|_{sup} \leq \varepsilon$

i.e. $|y(x) - \tilde{y}(x)| \leq \varepsilon \quad \forall x \in [c,d] = [a-h, a+h]$.

Theorem 1.2 (Gronwall's inequality)

Let $w: I \to [0,\infty)$ continuous be so that for $a \in I$, $B, C \geqslant 0$

(1) $w(x) \leq B + C \cdot \left|\int_a^x w(t)\,dt\right|$ $x \in I$.

Then: $w(x) \leq B \cdot e^{c|x-a|}$

Proof: $x \geqslant a$

$W(x) = \int_a^x w(t)\,dt \geqslant 0$ for $x \geqslant a$

$W(a) = 0$

Have

$W'(x) = w(x) \leq B + C\,W(x)$

$(e^{-cx} W(x))' = e^{-cx}(W'(x) - CW(x)) \leq B e^{-cx}$

$\Big| \int_a^x$

$e^{-cx} W(x) \leq -\frac{B}{c}\left(e^{-cx} - e^{-ca}\right)$

so $W(x) \leq \frac{B}{c} e^{c(x-a)} - \frac{B}{c}$

$\overset{(1)}{\Rightarrow}$ $w(x) \leq B + C \cdot W(x)$

$\leq B \cdot e^{c(x-a)}$ ▨

Application 1: Altern. proof of claim 4 in Picard proof = uniq.

Seen: $y, \tilde{y}$ solve same (IVP)

then $e(x) = y(x) - \tilde{y}(x)$

satisfies

$|e(x)| \leq L \cdot \int_a^x |e(t)|\,dt$

Set $w(t) = |e(x)|$ apply Gronwall

$(B = 0,\ C = L)$ get

$|e(x)| \leq 0 \cdot e^{\cdots} = 0$ ▨

Application 2: Continuous dependence:

Let $y, \tilde{y}$ be solutions of

(IVP), $(\widetilde{\text{IVP}})$

Then

$$y(x) - \tilde{y}(x) = b + \int_a^x f(t, y(t))\,dt - \left(\tilde{b} + \int_a^x f(t, \tilde{y}(t))\,dt\right)$$

$w(x) := |y(x) - \tilde{y}(x)|$ then

$$w(x) \le |b - \tilde{b}| + \left| \int_a^x \underbrace{|f(t, y(t)) - f(t, \tilde{y}(t))|}_{\le L \cdot w(t)}\,dt \right|$$

$$\le |b - \tilde{b}| + L \cdot \left| \int_a^x w(t)\,dt \right|$$

Gronwall $\rightarrow$

$$w(x) \le |b - \tilde{b}| \cdot e^{L \cdot |x-a|} \le |b - \tilde{b}| \cdot e^{L \cdot h}$$

so

$$\|y - \tilde{y}\|_{sup} \le |b - \tilde{b}| \cdot e^{Lh}$$

and given $\varepsilon > 0$ can choose

$\delta = e^{-Lh} \cdot \varepsilon$ to get

$$\|y - \hat{y}\|_{sup} \le \varepsilon \text{ if } |b - \hat{b}| \le \delta.$$