Recall:

Theorem 1.1 (Picard)

If $f : R = [a-h, a+h] \times [b-k, b+k] \to \mathbb{R}$

satisfies

(Pi) (a) f continuous, $|f(x,y)| \leq M$

(b) $Mh \leq k$

(Pii) Lip. cond

$$|f(x,y) - f(x,\tilde{y})| \leq L \cdot |y - \tilde{y}|$$

$$\forall x \in [a-h, a+h], y, \tilde{y} \in [b-k, b+k]$$

Then $\exists!$ solution of

$$(IVP) \begin{cases} y'(x) = f(x, y(x)) \\ y(a) = b \end{cases}$$

on $[a-h, a+h]$.

Today:

contraction mapping theorem $=$ CMT

1.6 Proof of Picard via CMT

Recall

Thm 1.3 (CMT)

Let (X,d) be complete metric space.

and assume that

$$T : X \to X$$

is a contraction, i.e. $\exists K < 1$

s.t.

$$d(Tx, Ty) \leq K \cdot d(x,y) \quad x, y \in X$$

Then: $\exists!\, y \in X$ s.t. $y = Ty$.

As (IVP) is equivalent to

$$(IE) \quad y(x) = b + \int_a^x f(t, y(t))\,dt$$

consider

$T : y \mapsto Ty$

$$(Ty)(x) := b + \int_a^x f(t, y(t))\,dt.$$

So $(IE) \iff y(x) = (Ty)(x)\ \forall x$

i.e. $y = Ty$.

Right space:

For f satisfying $(P_i), (P_{ii})$ we want

$$X = C([a-\eta, a+\eta], [b-k, b+k])$$

suitable $\eta \in (0, h]$.

⌈Note: If f satisfies (P_{iii})

($f : [a-h, a+h] \times \mathbb{R} \to \mathbb{R}$ cont & glob Lip)

take

$X = C([a-\eta, a+\eta], \mathbb{R})$⌋

Metric spaces:

Take $d(y, \tilde{y}) := \|y - \tilde{y}\|_{sup}$

$$= \sup_{x \in [a-\eta, a+\eta]} |y(x) - \tilde{y}(x)|$$

makes (X, d) into complete metric space.

<u>Thm 1.4</u> (Picard)

Let $f : R \to \mathbb{R}$ s.t. $(P_i)(a), (P_{ii})$ are satisfied. Let $0 < \eta \leq h$ be s.t.

$L\eta < 1$ and $M \cdot \eta \leq k$.

<u>Then</u>: (IVP) has a unique solution on $[a-\eta, a+\eta]$.

⌈e.g. $\eta < \min\{h, \frac{1}{L}, \frac{k}{M}\}$ will work⌋.

To check:

1. $T : X \to X$ ($\hat{=}$ claim 1)
2. T contraction ($\hat{=}$ ... 2)

① • Ty continuous if y continuous by contin. of f and FTC

• $|Ty(x) - b| \overset{\triangle}{\leq} \left| \int_a^x |f(t,y(t))| dt \right|$

$\leq M \cdot |x-a| \leq M \cdot \eta$

$\leq k$ ✓

i.e. $T: X \to X$.

② T contraction with $K = L \cdot \eta < 1$.

Let $y, \tilde{y} \in C([a-\eta, a+\eta], [b-k, b+k])$

To show
$\|Ty - T\tilde{y}\|_{sup} \leq K \cdot \|y - \tilde{y}\|_{sup}$.

i.e. show $\forall x \in [a-\eta, a+\eta]$

$|(Ty)(x) - (T\tilde{y})(x)| \leq$ - - - -

Note

$$|(Ty)(x) - (T\tilde{y})(x)| \leq \left| \int_a^x \underbrace{|f(t,y(t)) - f(t,\tilde{y}(t))|}_{\underset{Lip}{\leq} L \cdot |y(t) - \tilde{y}(t)| \leq L \cdot \|y - \tilde{y}\|_{sup}} dt \right|$$

$\leq |x-a| \cdot L \cdot \|y - \tilde{y}\|_{sup}$

$\leq L \cdot \eta \quad \|y - \tilde{y}\|_{sup}$

So assumpt. of CMT are satisfied: Hence $\exists! y \in X$ s.t.

$Ty = y$ i.e. s.t. (IE) is satisfied

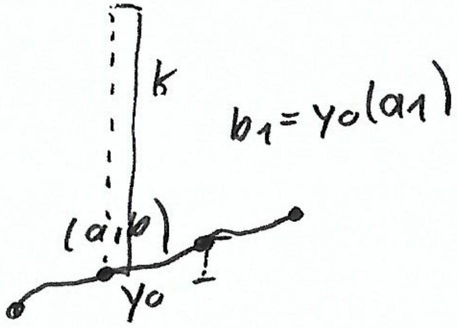
i.e. s.t (IVP) - - - -

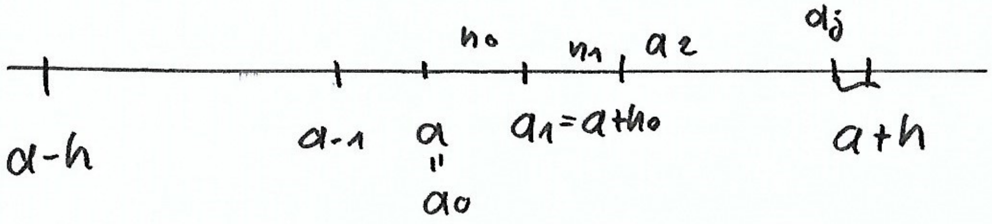
Using iteration we get

$Mh \leq k$

<u>Cor 1.5:</u>
If f satisfies (P_i) and (P_{ii}) then $\exists!$ sol. y of (IVP) on $[a-h, a+h]$.

Idea of proof:





$Lh_0 < 1$

Apply Thm 1.4 on rectangles

$$R_j = [a_j - h_j, a_j + h_j] \times [b_j - k_j, b_j + k_j]$$

chosen s.t.

- $a_{j+1} = a_j + h_j$
- $b_{j+1} = y_j(a_{j+1})$

For this to work we need:

- $R_j \subseteq R$
- $L h_j < 1$
- $M \cdot h_j \leq k_j$

L, M s.t. (Pi), (Pii) ok on $R = [a-h, a+h] \times [b-k, b+k]$.

- we reach "endpoint" $a+h$ resp $a-h$ after finitely many steps. (we'll finish after $\leq \left\lfloor \frac{h}{h_0} \right\rfloor + 1$ steps in either direction)

More precisely:

If $Lh < 1$ → apply Thm 1.4 once → done.

Else: **Fix** $h_0 > 0$ s.t. $Lh_0 < 1$. Thm 1.4 to get y_0 on $[a_0 - h_0, a_0 + h_0]$ define $a_0 = a$, $b_0 = b$

$$a_1 = a_0 + h_0, \quad b_1 = y_0(a_0 + h_0) = y_0(a_1).$$

Note: $|b_{x_j} - b| \leq M \cdot |a_{x_j} - a_0|$

so can choose

$h_{x_j} = \min(h_0, a+h - a_{x_j})$

$k_{x_j} = k - M \cdot |a_{x_j} - a_0|$

$\to R_{x_j} \subseteq R$.

Note: $L h_{x_j} \leq L h_0 < 1$

$M \cdot h_{x_j} \leq M \cdot h - M |a_{x_j} - a|$

$\leq k - M |a_{x_j} - a|$

$= k_{x_j}$

$\to$ Thm 1.4 applies and gives

y_1 on $[a_1 - h_1, a_1 + h_1]$

Can join up with y_0 to get a sol.

on $[a, a_2]$, $a_2 = a_1 + h_1$.

$b_2 = y_1(a_2)$

Note: Step size $= h_0$ except possibly in the last step (in each direct)

$\to$ process ends and gives us sol. on $[a-h, a+h]$

after $\leq \lfloor \frac{h}{h_0} \rfloor + 1$ step (per dir.)