Last time:

Proof of Picard via CMT

(IVP) equiv. to $Ty = y$

for $(Ty)(x) = b + \int_a^x f(t, y(t))\,dt$

- $T: X \to X$ if $M\eta \le k$
- T contraction if $L\eta < 1$

where

$X = C([a-\eta, a+\eta], [b-k, b+k])$,

$\eta \le h$.

<u>Remark</u>: If $f: [c,d] \times \mathbb{R} \to \mathbb{R}$ is continuous & satisfies (Piii) then work on

$X = C([a-\eta, a+\eta], \mathbb{R})$

so $T: X \to X$ ok just by Analysis III.

T contraction if $L \cdot \eta < 1$.

$\rightsquigarrow \exists!$ sol.
- initially on $[a-\eta, a+\eta]$

using iteration
- on $[c,d]$

(resp. if this holds on every bounded interval $\to$ get sol. $\forall x \in \mathbb{R}$).

1.7 Picard's Theorem for systems and for higher order ODEs

$$(\text{IVP}) \begin{cases} y_1'(x) = f_1(x, y_1(x), y_2(x)) \\ y_2'(x) = f_2(x, y_1(x), y_2(x)) \\ y_1(a) = b_1, \quad y_2(a) = b_2 \end{cases}$$

$$\underline{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} : I \to \mathbb{R}^2$$

$$\underline{f} : I \times \underset{\substack{\uparrow \\ \text{closed disc around} \\ (b_1, b_2) \text{ with rad. } k}}{2D\text{-set}} \to \mathbb{R}^2$$

where we measure distance between two points $\underline{y}, \tilde{\underline{y}} \in \mathbb{R}^2$

$$\|\underline{y} - \tilde{\underline{y}}\|_1 = |y_1 - \tilde{y}_1| + |y_2 - \tilde{y}_2|$$

so consider

$$B_k(\underline{b}) = \{ \underline{y} \in \mathbb{R}^2 : \underbrace{\|\underline{y} - \underline{b}\|}_{= |y_1 - b_1| + |y_2 - b_2|} \le k \}$$

. We ask that

(Hi) $\underline{f} : [a-h, a+h] \times B_k(\underline{b}) \to \mathbb{R}^2$ is continuous and

$$\|\underline{f}(x, \underline{y})\|_1 \le M \quad \forall x \in [a-h, a+h] \quad \forall \underline{y} \in B_k(b)$$

(Hii) $f_{1,2}$ satisfy a Lip cond.
so $\exists L_{1,2}$ s.t.

$$|f_{1,2}(x, \underline{y}) - f_{1,2}(x, \tilde{y})| \le L_{1,2} \|\underline{y} - \tilde{\underline{y}}\|$$

$$x \in [a-h, a+h]$$

or equiv. $\quad \underline{y}, \tilde{\underline{y}} \in B_k(b)$

$(\widetilde{\text{Hii}})$ $\exists L$ s.t.

$$\|f(x, \underline{y}) - f(x, \tilde{y})\|_1 \le L \cdot \|\underline{y} - \tilde{y}\|$$

Thm. 1.6:

Suppose $\underline{f}: [a-h, a+h] \times B_k(\underline{b}) \to \mathbb{R}^2$ satisfies (Hi) and (Hii).

Then $\exists\ \eta \in (0, h]$ s.t.

$\exists!$ solution y of (IVP) on $[a-\eta, a+\eta]$.

Here any η s.t. $\eta \le h$, $M \cdot \eta \le k$ will work.

Finally: If consider higher order ODES, e.g. 2^{nd} order

$$y''(x) = F(x, y(x), y'(x))$$

$$(y: I \to \mathbb{R})$$

$$y(a) = b_1, \quad y'(a) = b_2$$

$$\left[\ G(x, y(x), y'(x), y''(x)) = 0\ \right]$$

Define $y_1(x) = y(x)$

$y_2(x) = y'(x)$

we get that problem is equivalent to $\underline{y}(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$ satisfying

$$\underline{y}'(x) = \underline{f}(x, \underline{y}(x))$$

where

$$\underline{f}(x, (y_1, y_2)) = \begin{pmatrix} y_2 \\ F(x, (y_1, y_2)) \end{pmatrix}$$

$$\underline{y}(a) = \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

→ can treat this as a system of 1^{st} order ODES and if F satisfies a Lip cond can check that $\underline{f}$ sat. a Lip cond.

In particular: If y satisfies a Linear 2^{nd} order ODE then F and thus $\underline{f}$ satisfy a GLOBAL Lip cond.

→ get solutions $\forall x$ where F is def.

- solutions of (IVP) might be non-unique, might not exist $\forall x$.

- Picard's thm.

  Suit. cond. on $f : [a-h, a+h] \times [b-k, b+k] \to \mathbb{R}$

  guarantee results on existence, uniqueness, cont. dependence of solutions of ODEs/IVPs.

- Gronvall's inequality

  For (non-neg.) functions we can turn a integral inequality into bound for the function.