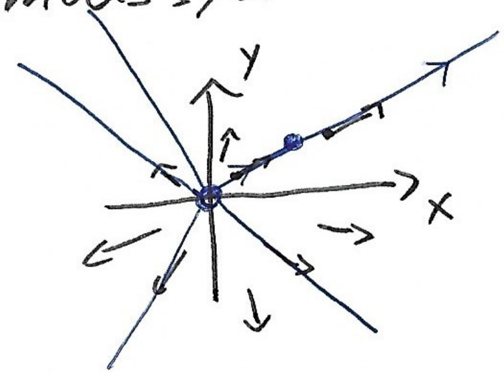
Chapter 2 Plane autonomous systems of ODEs:

Goal: Describe properties of solutions of

$$(1)\quad \begin{cases} \dot{x}(t) = X(x(t), y(t)) \\ \dot{y}(t) = Y(x(t), y(t)). \end{cases}$$

Note: X, Y are not allowed to explicitly depend on time:
That's why we call (1) an autonomous system.

Ex: $\dot{x}(t) = x(t)$
$\dot{y}(t) = y(t)$



Note: We are interested in the "paths" that solutions of (1) trace out/follow in xy-plane = trajectories.

Our assuption will always be that $X, Y: \mathbb{R}^2 \to \mathbb{R}$ are Lipschitz continuous (on every bd set)

Remark: Because our system is autonomous we have:

If $(x(t), y(t))$ of (1) then for any $T \in \mathbb{R}$ fixed also $(x(t+T), y(t+T))$ will satisfy (1)

e.g. $\dot{x}(t+T) = X(x(t+T), y(t+T))$

Consequence:

Through every point (x_0, y_0) $\exists!$ trajectory of (1)

Reason: By Picard $\exists!$ sol. $(x(t), y(t))$ of (1) s.t.

$$(x(0), y(0)) = (x_0, y_0).$$

And: If $(\tilde{x}(t), \tilde{y}(t))$ is a sol. of (1) with $(\tilde{x}(t_0), \tilde{y}(t_0)) = (x_0, y_0)$ for some t_0

then $(x(t+t_0), y(t+t_0))$ is also sol. of (1) and satisfies same init. cond. as $(\tilde{x}, \tilde{y})$ so by Picard:

$$(\tilde{x}, \tilde{y})(t) = (x, y)(t+t_0)$$

2.1. Critical points and closed trajectories

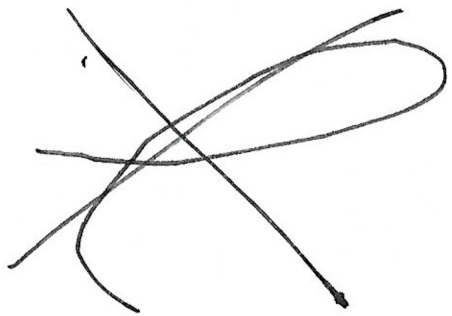
A point (x_0, y_0) is a critical point of (1) if $X(x_0, y_0) = 0$ and $Y(x_0, y_0) = 0$.

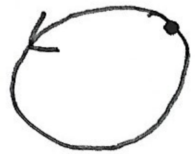
For critical points we have

$$(x(t), y(t)) = (x_0, y_0)$$ is a solution of (1).

We might also get closed trajectories coming from a solution $(x(t), y(t))$ for which we have $\exists t_1 \neq t_2$ s.t.

$$(x, y)(t_1) = (x, y)(t_2)$$





Note that if

$$(x,y)(t_1) = (x,y)(t_2)$$

for some $t_1 \neq t_2$ then for $T = t_2 - t_1$ we have

$$(x,y)(t+T) = (x,y)(t).$$

<u>Reason:</u>

$$(x,y)(t+T)\Big|_{t=t_1} = (x,y)(t_2) = (x,y)(t_1)$$

so

$(x,y)(t)$, $(x,y)(t+T)$ solve (1) & same value at t_1

$\Rightarrow$ must coincide.

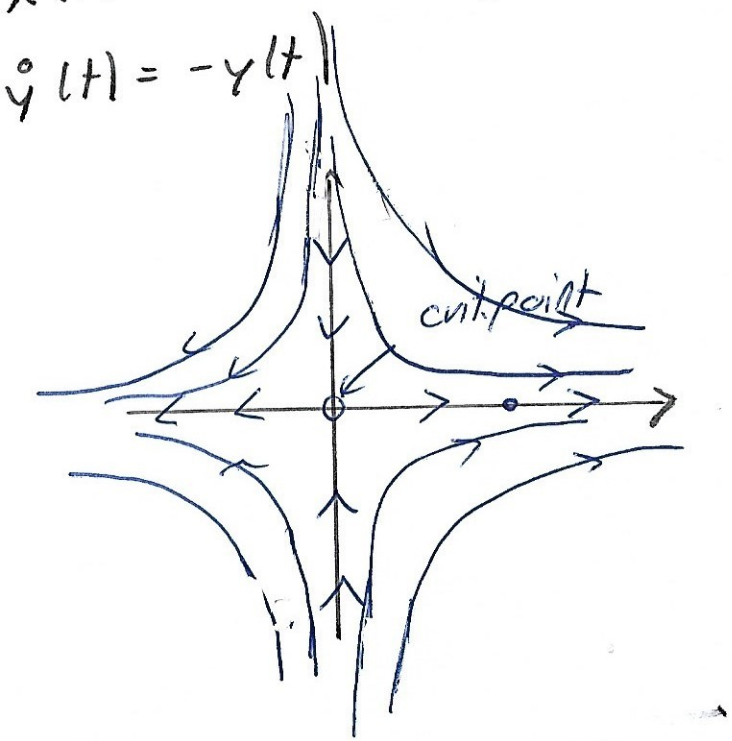
So such a solution must be periodic & corresp. traj. is closed.

<u>Also:</u> Two trajectories cannot really intersect but they can approach (as $t \to \pm\infty$) the same point and any such point will be a critical point.

<u>Ex.:</u>

$$\dot{x}(t) = x(t)$$
$$\dot{y}(t) = -y(t)$$

$$(x(t), y(t)) = (x_0 e^{t}, y_0 e^{-t}$$

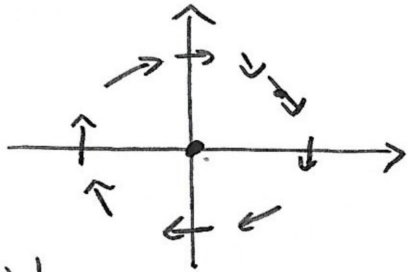


Ex: harmonic oscillator

$$\ddot{x}(t) = -\omega^2 x(t)$$

Set $y(t) = \dot{x}(t)$

$$\rightarrow \text{get} \begin{cases} \dot{x} = y = X(x,y) \\ \dot{y} = -\omega^2 x = \cancel{X}\, Y(x,y) \end{cases}$$



conserved quantity

$$\frac{d}{dt}\left(\omega^2 x^2(t) + y^2(t)\right)$$
$$= 2\omega^2 x(t)\,\dot{x}(t) + 2y(t)\,\dot{y}(t)$$
$$= 0$$

so along each solution we have

$$\omega^2 x^2(t) + y^2(t) \equiv \text{const} = c^2$$

my trajectory is an ellipse

$$\frac{x^2}{\left(\frac{c}{\omega}\right)^2} + \frac{y^2}{c^2} = 1$$

centred at $(0,0)$ halfaxis $\frac{c}{\omega}, c$, $c \geq 0$

with clockwise orientation.