2. Plane autonomous systems

$$(1)\begin{cases} \dot{x}(t) = X(x(t), y(t)) \\ \dot{y}(t) = Y(x(t), y(t)) \end{cases}$$

<u>Goal</u>: Qualitative information on solutions, in particular want to be able to draw "phase diagrams" = diagrams depicting trajectories (= paths of solutions) in $\mathbb{R}^2$ (= "phase plane")

<u>Now</u>:
Behaviour of solutions to (1) "near" critical points.

- stability
- Linearisation

] sect. 2.2

- classification of crical points

] sect. 2.3 → next video

2.2. Stability & Linearisation:

Let (a,b) critical point of (1)

$$X(a,b) = 0 = Y(a,b).$$

(Know $(x(t), y(t)) = (a,b)$ $\forall t$ solves (1))

<u>Q</u>: If we start close to (a,b) do we remain close to (a,b) for all times?

Def: A critical point (a,b) is stable

if $\forall \varepsilon > 0 \ \exists \delta > 0$, t_0

s.t. any solution $(x(t), y(t))$ of (1)

with $\sqrt{(x(t_0)-a)^2 + (y(t_0)-b)^2} < \delta$

then $\forall t \geq t_0$ we have

$$\sqrt{(x(t)-a)^2 + (y(t)-b)^2} < \varepsilon.$$

Note: If (a,b) is not stable we call it an unstable critical point.

Remark: Can just take $t_0 = 0$ since translations of solutions by fixed time will again be solutions since our (1) is autonomous.

To analyse solutions/trajectories near critical points the idea is to linearise equation and use that for LINEAR systems

$$\dot{\underline{z}}(t) = M \cdot \underline{z}(t)$$

we have explicit solutions, behaviour determined by eigenvalues of M.

Let (a,b) crit. point of (1)

Let $(x(t), y(t))$ solution of (1)

which is close to (a,b).

Write

$$x(t) = a + \xi(t)$$
$$y(t) = b + \eta(t)$$

ξ, η "small functions"

Then:

$$\mathring{\xi}(t) = \mathring{x}(t) = X(a+\xi(t), b+\eta(t))$$

$$= \underbrace{X(a,b)}_{=0} + X_x(a,b)\,\xi(t) + X_y(a,b)\,\eta(t) + \text{h.o.}$$

Similarly

$$\mathring{\eta}(t) = Y_x(a,b)\,\xi(t) + Y_y(a,b)\cdot\eta(t) + \text{h.o.}$$

so setting $\underline{\underline{z}}(t) = \begin{pmatrix} \xi(t) \\ \eta(t) \end{pmatrix}$ satisfies

$$\mathring{\underline{\underline{z}}}(t) = M(a,b)\circ \underline{\underline{z}}(t) + \text{h.o.}$$

where

$$M(x,y) = \begin{pmatrix} X_x(x,y) & X_y(x,y) \\ Y_x(x,y) & Y_y(x,y) \end{pmatrix}.$$

Expectation is that

$\underline{\underline{z}}(t) = \begin{pmatrix} x(t)-a \\ y(t)-b \end{pmatrix}$ behaves like a solution

of LINEAR system

$$(L) \quad \mathring{z} = M\cdot z$$

while/when $(x(t), y(t))$ close to (a,b).

Note that we need to ask that (a,b) is a "non-degenerate" critical point

i.e. $M(a,b)$ is invertible

or equiv. $\lambda = 0$ is NOT an eigenvalue!.

Solutions of LINEAR plane autonomous systems:

(L) $\mathring{\underline{z}}(t) = \underbrace{\begin{pmatrix} A & B \\ C & D \end{pmatrix}}_{= M \leftarrow \text{ any } 2\times 2 \text{ matrix}} \underline{z}(t)$

<u>Note</u>: If λ is an eigenvalue

$\underline{z_0}$ .... eigenvector

then $c_0 \cdot \underline{z_0} e^{\lambda t}$ is a sol. of (L)

• If eigenvalues λ_1, λ_2 are distinct,

i.e. $\lambda_1 \neq \lambda_2$, then general solution of (L) is

$$\underline{z}(t) = c_1 \cdot \underline{z_1} e^{\lambda_1 t} + c_2 \underline{z_2} e^{\lambda_2 t}$$

corresp. eigenvectors.

If $\lambda_{1,2} \in \mathbb{R} \rightarrow \underline{z_1}, \underline{z_2}, c_1, c_2$ all real

If $\lambda_1 \notin \mathbb{R} \Rightarrow \lambda_2 = \overline{\lambda_1}$

$\underline{z_2} = \overline{\underline{z_1}}$

so need $c_1 \in \mathbb{C}$, c_2 to be s.t. $c_2 = \overline{c_1}$

so get

$$\underline{z}(t) = 2 \operatorname{Re}(c_1 \underline{z_1} e^{\lambda_1 t})$$

$\lambda_1 = \lambda_2 = \lambda$: $\quad (\lambda \in \mathbb{R})$

Case 1: $\quad M = \lambda \cdot I$

$\rightarrow$ every vector $\underline{z_0} \neq 0$ is an eigenvector

$\rightarrow$ general solution

$$z(t) = \underline{z_0} \cdot e^{\lambda t}, \quad \underline{z_0} \in \mathbb{R}^2.$$

Case 2: $\quad \lambda_1 = \lambda_2 = \lambda$

$$M \neq \lambda I$$

so $\exists\ \underline{z_1}$ s.t. $(M - \lambda I)\, \underline{z_1} \neq 0$

Cayley-Hamilton thm.

$$(M - \lambda I)^2 = 0$$

So setting

$$\underline{z_0} = (M - \lambda I)\, \underline{z_1}$$

we get

$$(M - \lambda I)\, \underline{z_0} = \underbrace{(M - \lambda I)^2}_{=0} \underline{z_1} = 0.$$

General solution of (4) in this case is

$$z(t) = (c_1 \underline{z_1} + (c_0 + c_1 t)\, \underline{z_0})\, e^{\lambda t}.$$