Last video:

Linearisation of

$$(1)\begin{cases} \dot{x}(t) = X(x(t), y(t)) \\ \dot{y}(t) = Y(x(t), y(t)) \end{cases}$$

around critical point (a,b).

If $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \underline{z}(t)$, $\underline{z}(t)$ small,

then upto higher order terms

$$(L) \quad \dot{\underline{z}}(t) = M(a,b) \cdot \underline{z}(t)$$

$$M(a,b) = \begin{pmatrix} X_x & X_y \\ Y_x & Y_y \end{pmatrix}_{(a,b)}$$

Today:

Classify critical points based on properties of eigenvalues λ_1, λ_2 of $M(a,b)$.

Assumption: Critical point non-degenerate, i.e. $\lambda_1, \lambda_2 \neq 0$.

2.3 Classification of critical points

(L) $\dot{z} = M \cdot z$

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

$A, B, C, D \in \mathbb{R}$.

eigenvalues of M

λ_1, λ_2

Recall if $\lambda_1 \neq \lambda_2$ then

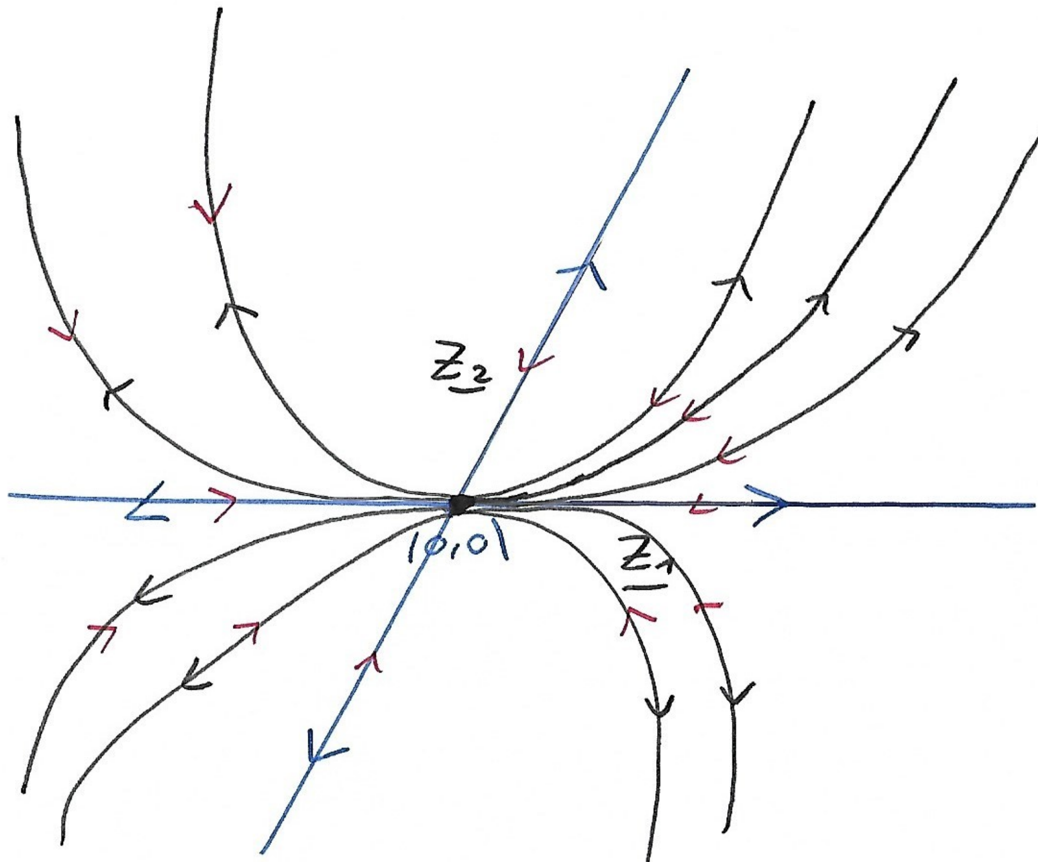
$$z(t) = c_1 \underline{z_1} e^{\lambda_1 t} + c_2 \underline{z_2} e^{\lambda_2 t}$$

eigenvectors

Case 1: $0 < \lambda_1 < \lambda_2$ "unstable node"

$z(t) \to \quad t \to -\infty$

1st term $\to 0$ slower than 2nd term.



$t \to \infty$: Both terms $\to \infty$

2nd $\to \infty$ faster

Case 2: $\lambda_2 < \lambda_1 < 0$ "stable node"

If $z(t)$ solves (L)

$\to$ $\tilde{z}(t) = z(-t)$ solves (L) but with $-M$ $\quad (-\lambda_2 > -\lambda_1 > 0)$

Case 3: $\lambda_1 = \lambda_2 = \lambda$:

Case 3a: $M = \lambda I$.

"stable or unstable star"

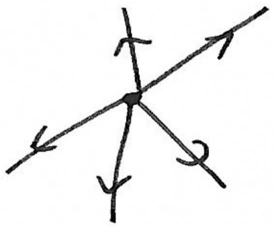
$\lambda < 0$ (stable), $\lambda > 0$ (unstable)

$\dot{x} = \lambda x$

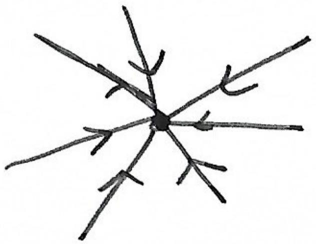
$\dot{y} = \lambda y$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \cdot e^{\lambda t}$$

$\lambda > 0$



$\lambda < 0$



Case 3b $\lambda_1 = \lambda_2 = \lambda$, $M \neq \lambda I$

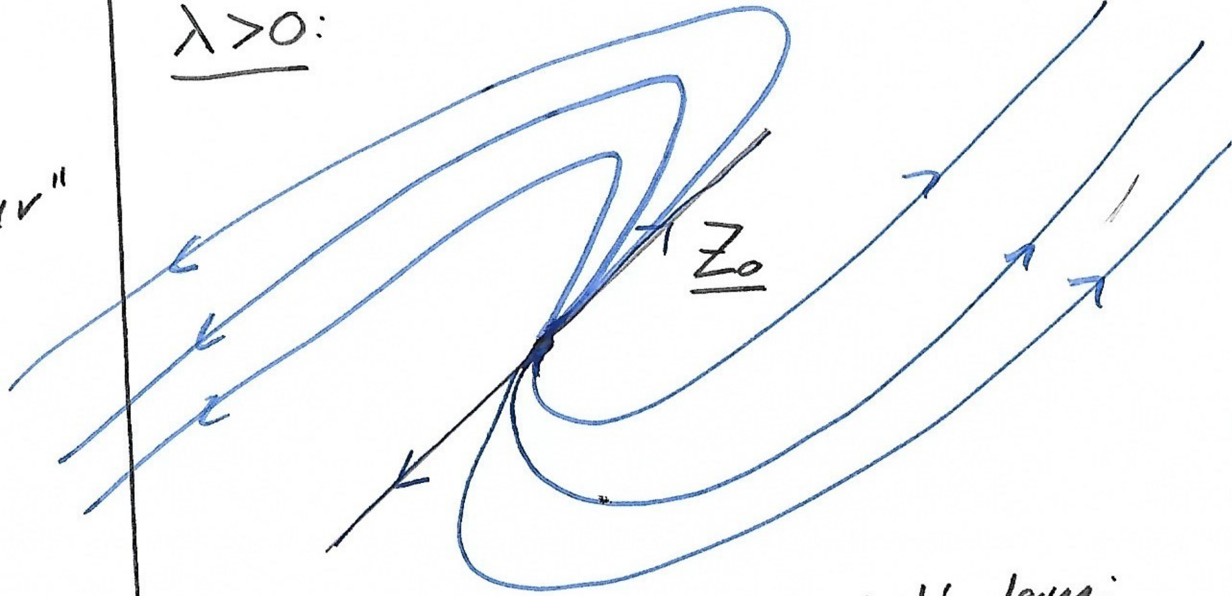
$$\underline{Z}(t) = (c_1 \underline{Z_1} + (c_0 + c_1 t) \cdot \underline{Z_0}) e^{\lambda t}$$

$\underline{Z_0}$ = eigenvector

$\lambda > 0$: unstable inflected node

$\lambda < 0$: stable -- --

$\lambda > 0$:



$c_1 > 0$, $t \to \infty$ dominating term:

$$c_1 t \, \underline{Z_0} \, e^{\lambda t}$$

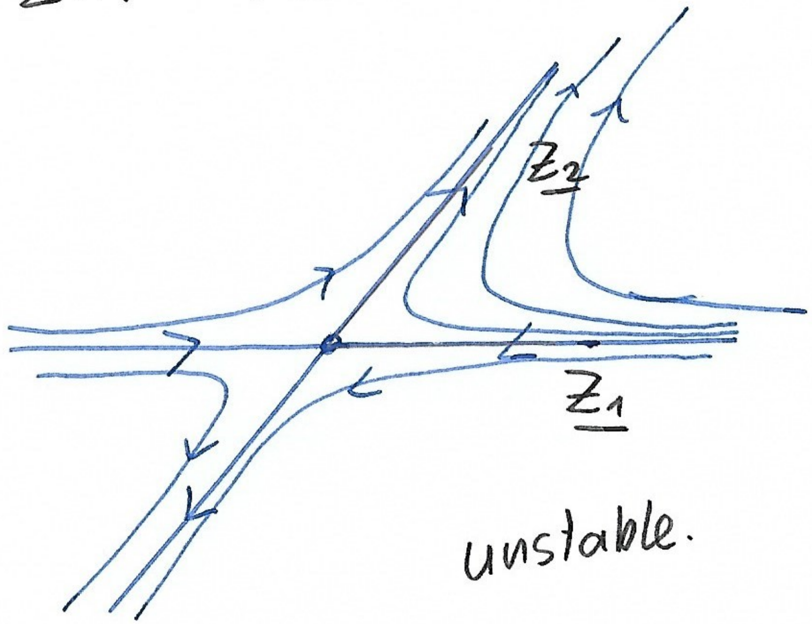
$t \to -\infty$ domin. term

$$\underbrace{c_1 t}\, \underline{Z_0} \, e^{\lambda t}$$

different sign!

Case 4: $\lambda_1 < 0 < \lambda_2$: "saddle"

$$\underline{z}(t) = c_1 \underline{z}_1 e^{\lambda_1 t} + c_2 \underline{z}_2 e^{\lambda_2 t}$$



unstable.

Complex eigenvalues:

$\lambda_1 = \mu + i\omega \quad , \lambda_2 = \mu - i\omega$

$\underline{z_1} = \begin{pmatrix} 1 \\ \kappa e^{i\phi} \end{pmatrix} \quad , \underline{z_2} = \overline{z_1} = \begin{pmatrix} 1 \\ \kappa e^{-i\phi} \end{pmatrix}$

$\kappa \in \mathbb{R}$

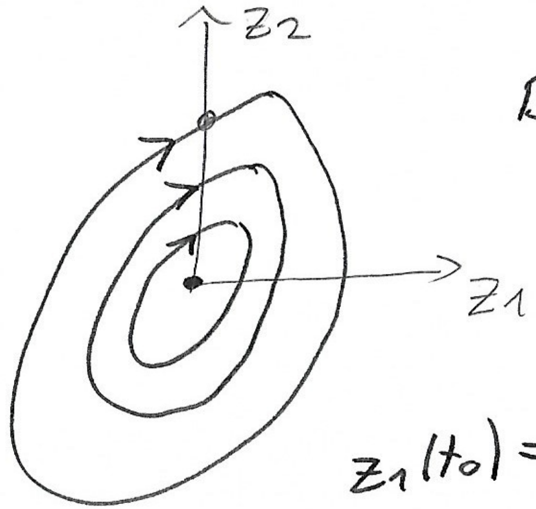
$c_1 = r \cdot e^{i\theta} \quad , c_2 = r e^{-i\theta}$

$$z(t) = 2 \, \mathrm{Re} \left(r \cdot e^{i\theta} \begin{pmatrix} 1 \\ \kappa e^{i\phi} \end{pmatrix} e^{i\omega t} \right) \cdot e^{\mu t}$$

$$= 2r \cdot \begin{pmatrix} \cos(\omega t + \theta) \\ \kappa \cos(\omega t + \theta + \phi) \end{pmatrix} e^{\mu t}$$

Case 5: $\mu = \mathrm{Re}(\lambda_1) = 0$ "centre"

$z(t)$ is periodic (periode $\frac{2\pi}{\omega}$)



$B > 0$

$z_1(t_0) = 0$

$\underline{\dot{z}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot z$

$\dot{z}(t_0) = \begin{pmatrix} B z_2(t_0) \\ D z_2(t_0) \end{pmatrix}$

If $B > 0$, $z_2(t_0) > 0$ go to right → clockwise

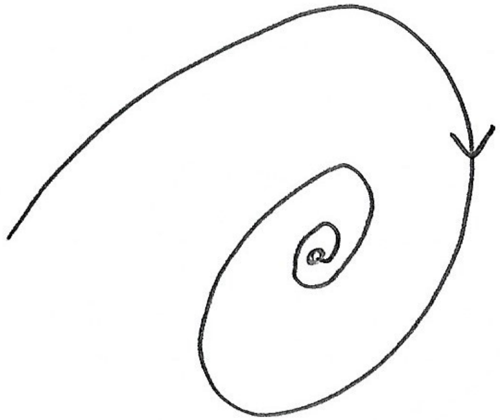
$B < 0 \quad z_2(t_0) < 0$ ... left → anticlockwise

Case 6: $\lambda_{1,2}$ complex conj.

$\mu = \mathrm{Re}(\lambda_1) \neq 0$

$\mu > 0$ unstable
$\mu < 0$ stable

Spiral in clockwise $B > 0$
anticlockw. $B < 0$



$\mu < 0$
$B > 0$

Remark:

$\mathrm{trace}(M) = A + D = \lambda_1 + \lambda_2$

$\det(M) = AD - BC = \lambda_1 \lambda_2$

If $\det(M) > 0$

- λ_1, λ_2 real
  - both same sign
    - node
    - star
    - inflected node

  and this is
  stable if trace < 0
  unstable if trace > 0.

- $\lambda_1 = \overline{\lambda_2} \in \mathbb{C} \setminus \mathbb{R}$

  $\det = |\lambda|^2$

  $\mathrm{trace}(M) = 2\mathrm{Re}(\lambda_1)$

  so
  - centre if $\mathrm{trace} M = 0$
  - unstable spiral trace > 0
  - stable spiral trace < 0.

If $\det M < 0$

then point = saddle

$\rightarrow$ unstable!

Remark:

Q: How reliable is this picture for behaviour of trajectories of the non-linear system?

Rough answer:

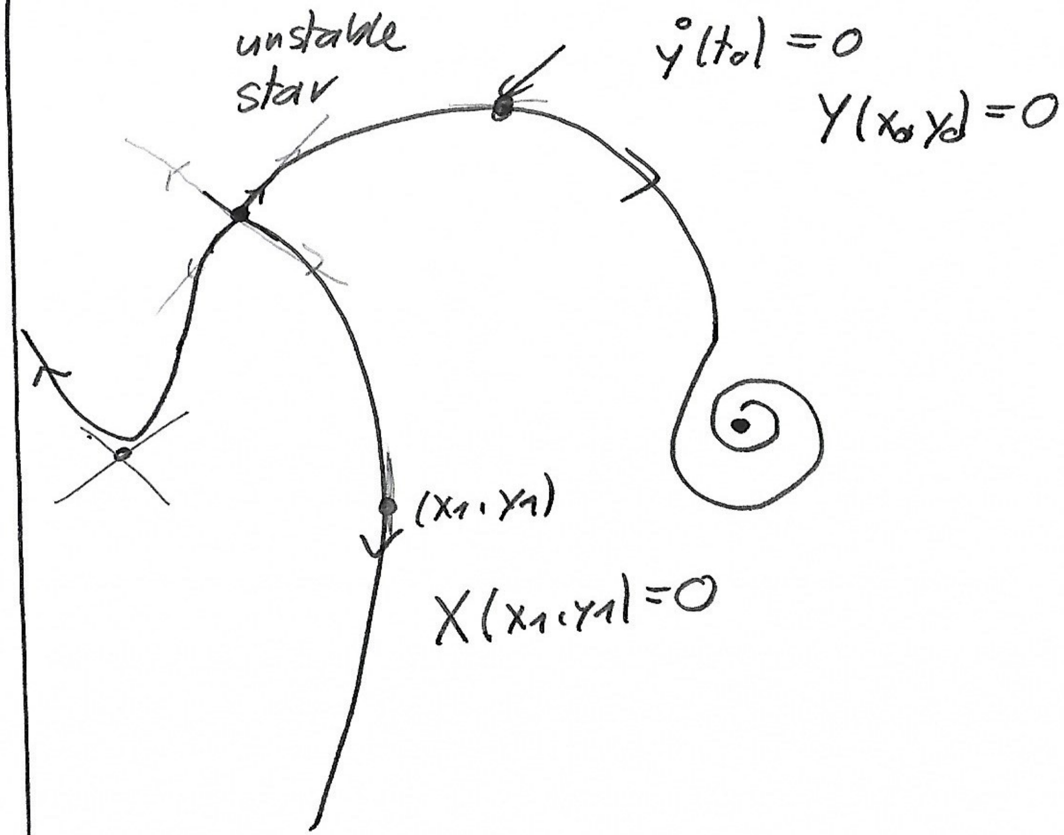
- only applies NEAR a critical point

However: Unless we have a centre

If linearisation has stable/unstable critical point

Then also the corresp. crit. point of nonlin system is unstable/stable.

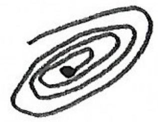
Also: Same type of critical point



⚠ For a centre we require $Re(\lambda)=0$

know any deviation from this can change stability

e.g. can get stable spiral "sprivals in slower" than for $Re(\lambda)<0$

.