Last time

Classification of critical points:

$\lambda_{1,2}$ real

- distinct, same sign: node
- equal $\begin{cases} \text{star} \\ \text{inflected node} \end{cases}$

stable if λ's < 0

un - - - - - > 0

- different signs $\Rightarrow$ saddle

complex conjugate pair:

- $\mathrm{Re}\,\lambda = 0$ $\rightarrow$ centre (periodic, ellipses)
- $\mathrm{Re}\,\lambda > 0$ unstable spiral
  < 0 stable spiral

$\rightarrow$ gives good picture near critical points of non-linear systems

Today:

How to draw phase diagrams?

Remark:

Curves where $X(x,y) = 0$

resp. $Y(x,y) = 0$

are the ONLY place where trajectories can and will become vertical resp. horizontal as there we have

$$\dot{x}(t_0) = X(x(t_0), y(t_0)) = 0$$

resp

$$\dot{y}(t_0) = 0.$$

These curves are called nullclines, these intersect at critical points.

Ex. 1:

$$\dot{x} = x - y = X(x,y)$$

$$\dot{y} = 1 - xy = Y(x,y)$$

critical points:

$x = y$ and $1 = xy$

$p_1 = (1,1)$, $p_2 = (-1,-1)$.

$$M(x,y) = \begin{pmatrix} 1 & -1 \\ -y & -x \end{pmatrix}$$

$$M(1,1) = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}, \quad \chi(\lambda) = (1-\lambda)(-1-\lambda) - 1 = \lambda^2 - 2$$

so $\lambda_1 = -\sqrt{2}$, $\underline{z_1} = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$

$\lambda_2 = \sqrt{2}$, $\underline{z_2} = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$

saddle

$$M(-1,-1) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\chi(\lambda) = (1-\lambda)^2 + 1 = 0$$

$\lambda_{1,2} = 1 \pm i$, $Re(\lambda_i) = 1 > 0$ unstable spiral

$B = -1 < 0$ anticlockwise

Nullclines:

$X(x,y) = 0$ "vertical"

$$\boxed{y = x}$$

$Y(x,y) = 0$ "horizontal"

$$\boxed{y = \frac{1}{x}}$$

$p_1 = (1,1)$ saddle

$p_2 = (-1,-1)$ spiral

$\lambda_1 < 0 \quad \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$

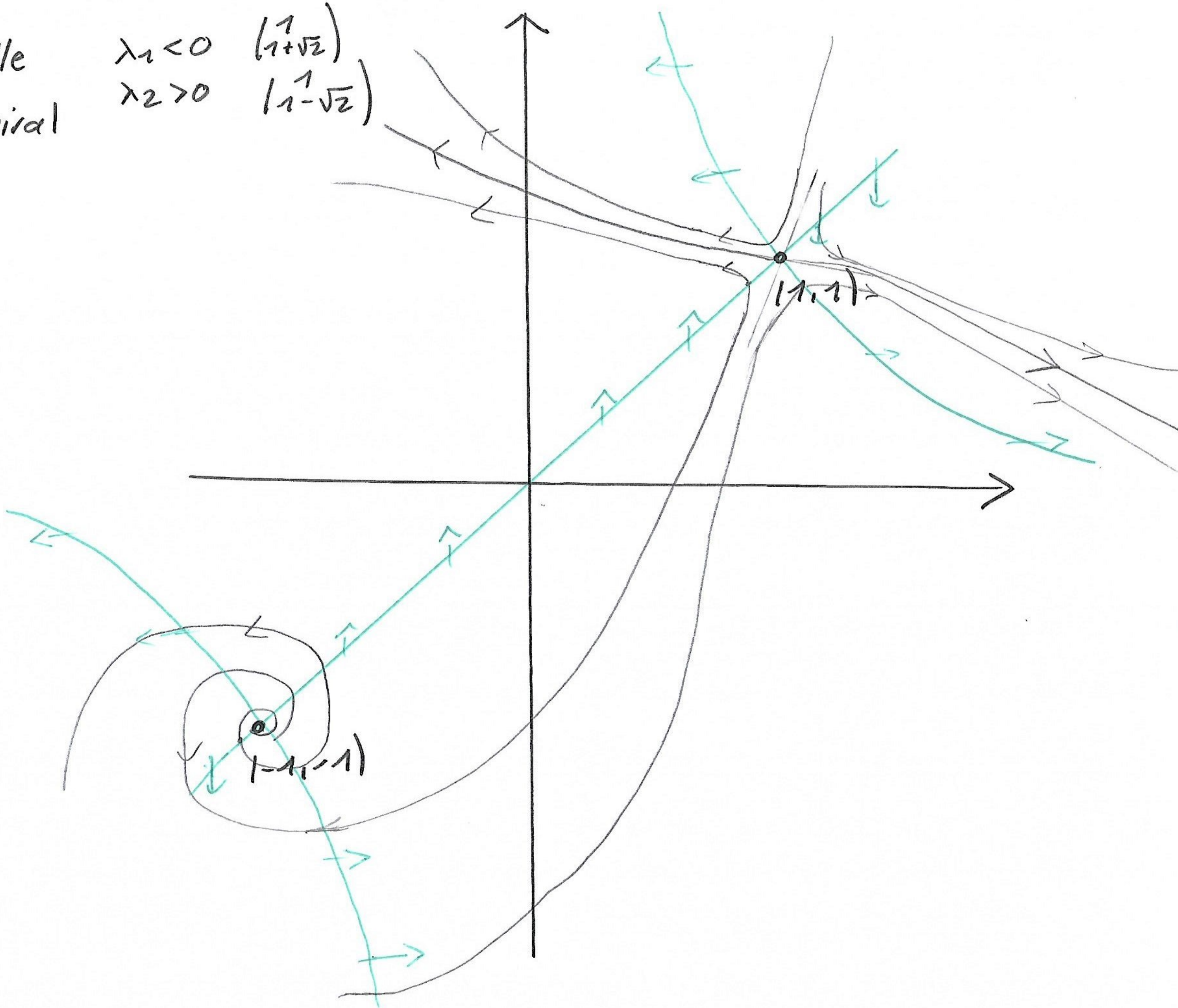
$\lambda_2 > 0 \quad \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$

$y = x \qquad X = 0$

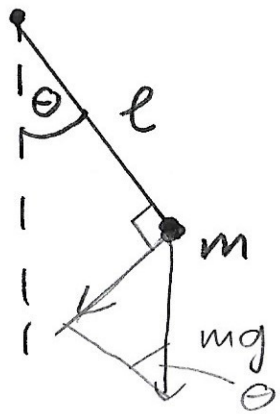
$y = \frac{1}{x} \qquad y = 0$

$X(x,y) = x - y$

$Y(x,y) = 1 - xy$



Ex 2 Dampened pendulum:



$$m\ell\cdot\ddot{\theta} = -mg\cdot\sin\theta - mk\ell\dot{\theta}$$

$k > 0$

Set $x = \theta$, $y = \dot{x}$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\frac{g}{\ell}\sin x - ky \end{cases}$$

crit. points: $y = 0$, $x = \begin{cases} 2\pi n, & n \in \mathbb{Z} \\ 2\pi n + \pi, & n \in \mathbb{Z}. \end{cases}$

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell}\cos x & -k \end{pmatrix}$$

even mult. of π:

$$M = \begin{pmatrix} 0 & 1 \\ -\frac{g}{\ell} & -k \end{pmatrix}$$

For $k^2 < \frac{4g}{\ell}$ eigenval. are complex conjugates

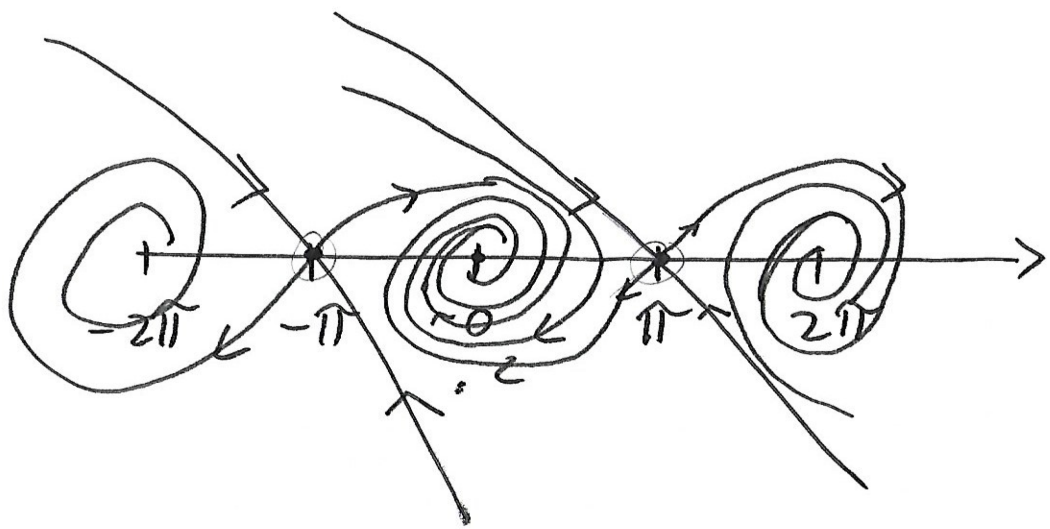
As trace$(M) = -k < 0$ → stable spiral

$1 > 0$ → clockwise

odd mult. of π: $M = \begin{pmatrix} 0 & 1 \\ \frac{g}{k} & -k \end{pmatrix}$

$\det(M) = -\frac{g}{k} < 0$ → saddle, $\lambda_1 < 0 < \lambda$

$M - \lambda_1 I = \begin{pmatrix} + & + \\ - & - \end{pmatrix}$ $z_1 = \begin{pmatrix} + \\ - \end{pmatrix}$, $M - \lambda_2 I = \begin{pmatrix} - & + \\ - & - \end{pmatrix}$ $z_2 = \begin{pmatrix} + \\ + \end{pmatrix}$



-2π
-π
0
π
2π