Examples from population dynamics:

<u>Example 3</u>: Lotka-Volterra predator-prey equations

x = population of pray

y = ... predator

$$\dot{x} = \alpha x - \gamma xy = X(x,y)$$

$$\dot{y} = -\beta y + \delta xy = Y(x,y)$$

$\alpha, \beta, \gamma, \delta > 0$

<u>crit. points</u> $X(x,y) = x \cdot (\alpha - \gamma y)$

$Y(x,y) = y \cdot (-\beta + \delta x)$

$p_1 = (0,0)$, $\left(\frac{\beta}{\delta}, \frac{\alpha}{\gamma}\right) = p_2$

$(0,0)$ $\quad M(x,y) = \begin{pmatrix} \alpha - \gamma y & -\gamma x \\ \delta y & -\beta + \delta x \end{pmatrix}$

$$M(0,0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\beta \end{pmatrix}$$

$\lambda_1 = \alpha > 0$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = -\beta < 0$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

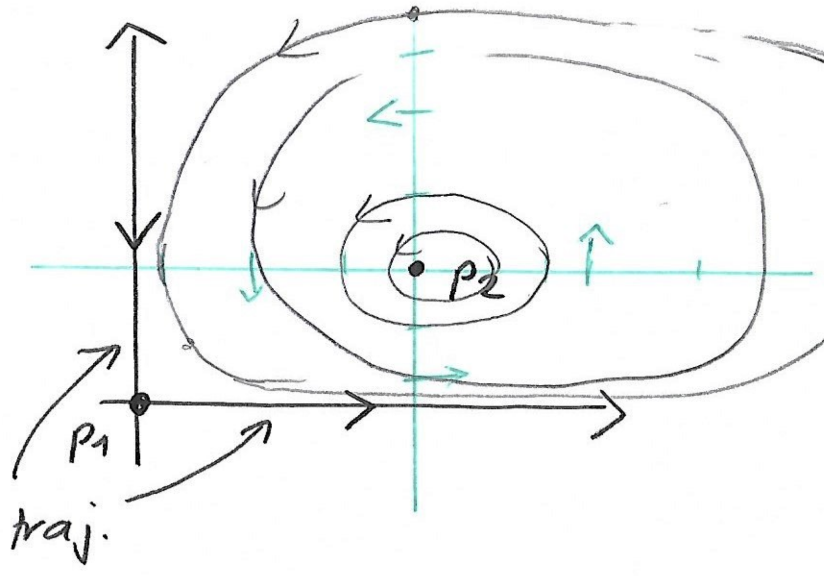
saddle

$$M\left(\frac{\beta}{\delta}, \frac{\alpha}{\gamma}\right) = \begin{pmatrix} 0 & -\gamma\beta/\delta \\ \frac{\delta\alpha}{\gamma} & 0 \end{pmatrix}$$

$$\lambda^2 + \alpha\beta = 0$$

$$\boxed{\lambda = \pm i\sqrt{\alpha\beta}}$$

centre!



$\dot{x} = x(\alpha - \gamma y)$

$\dot{y} = y(-\beta + \delta x)$

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{y}{\alpha - \gamma y} \cdot \frac{-\beta + \delta x}{x}$$

$$\left(\frac{\alpha}{y} - \gamma\right) dy = \left(-\frac{\beta}{x} + \delta\right) dx$$

$$\alpha \log(y) - \gamma y = -\beta \log(x) + \delta x + C$$

trajectories are sets

$$\{ (x,y) \in \mathbb{R}^2 : \quad f(x,y) = C \}$$

where $f(x,y) = \beta \log x - \delta x + \alpha \log y - \gamma y$

f has maximum is at p_2 (corresp. level set = traj. = 1 point)

Can't have spirals (can't approach a maximum by level sets with a different, fixed C)

<u>Note:</u> $f(x,y) \to -\infty$

as $x \to 0$ or $y \to 0$

$x \to \infty$ or $y \to \infty$

<u>All</u> trajectories (1st quadrant) are periodic.

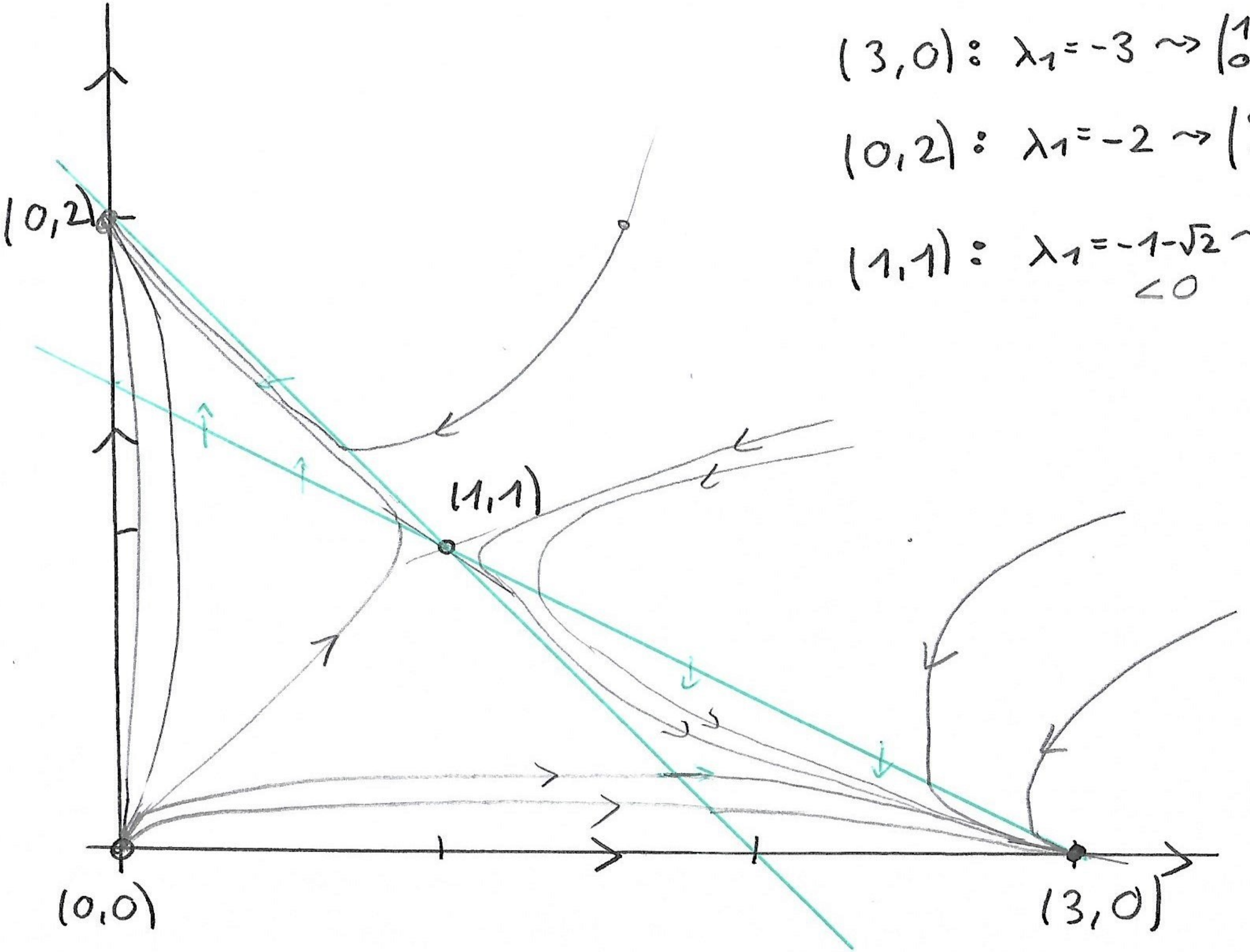
Example 4: Competing species:

x = population of rabits

y = - - - - - . sheep

$$\begin{cases} \dot{x} = (3-x-2y)\cdot x \\ \dot{y} = (2-x-y)\cdot y \end{cases}$$

$\dot{x} = (3-x-2y)x, \quad \dot{y} = (2-x-y)y$



$(0,0)$: $\lambda_1 = 3 \rightsquigarrow \begin{pmatrix}1\\0\end{pmatrix}$, $\lambda_2 = 2 \rightsquigarrow \begin{pmatrix}0\\1\end{pmatrix}$

$(3,0)$: $\lambda_1 = -3 \rightsquigarrow \begin{pmatrix}1\\0\end{pmatrix}$, $\lambda_2 = -1 \rightsquigarrow \begin{pmatrix}-3\\1\end{pmatrix}$

$(0,2)$: $\lambda_1 = -2 \rightsquigarrow \begin{pmatrix}0\\1\end{pmatrix}$, $\lambda_2 = -1 \rightsquigarrow \begin{pmatrix}-1\\2\end{pmatrix}$

$(1,1)$: $\underbrace{\lambda_1 = -1-\sqrt{2}}_{<0} \rightsquigarrow \begin{pmatrix}\sqrt{2}\\1\end{pmatrix}$, $\lambda_2 = -1+\sqrt{2} > 0 \rightsquigarrow \begin{pmatrix}-\sqrt{2}\\1\end{pmatrix}$

<u>Either</u> sheeps or rabbits survive