This video:

Periodic solutions

- Example 5: limit circle
- 2.4 Bendixson-Dulac Theorem

Ex. 5:

$$\dot x = (1-\sqrt{x^2+y^2})x - y$$

$$\dot y = (1-\sqrt{x^2+y^2})\cdot y + x$$

Critical point: $(0,0)$

$$y = (1-r)x = -(1-r)^2\cdot y$$

$$r = \sqrt{x^2+y^2}$$

$M = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \to$ unstable spiral anticlockwise.

Try polar coord.:

$$x = r\cdot\cos\theta$$

$$y = r\cdot\sin\theta$$

$$2r\dot r = \dot{(r^2)} = \dot{(x^2+y^2)} = 2x\dot x + 2y\dot y$$

$$= 2x^2(1-r) - \cancel{2xy}$$

$$+2y^2(1-r) \cancel{+ 2xy}$$

$$= 2r^2(1-r)$$

$$\boxed{\dot r = r(1-r)}$$

$$\cancel{(1-r)x} - y = \dot x = \dot{(r\cdot\cos\theta)} = \overset{r(1-r)}{\cancel{\dot r}}\cos\theta - r\cdot\sin\theta\cdot\dot\theta$$

$$-r\sin\theta = -r\sin\theta\cdot\dot\theta$$

$$\boxed{\dot\theta = 1}$$

$$\begin{cases} \dot{r} = r(1-r) \\ \dot{\theta} = 1 \end{cases} \qquad \to \theta(t) = t + \theta_0$$

$$\int \frac{1}{\underbrace{r(1-r)}} \, dr = \int dt = t + c$$

$$= \frac{1}{r} + \frac{1}{1-r}$$

$$\log(r) - \log(|1-r|) = t + c$$

$$\frac{r}{|1-r|} = e^c \cdot e^t$$

$$\frac{r(t)}{1-r(t)} = A \cdot e^t$$

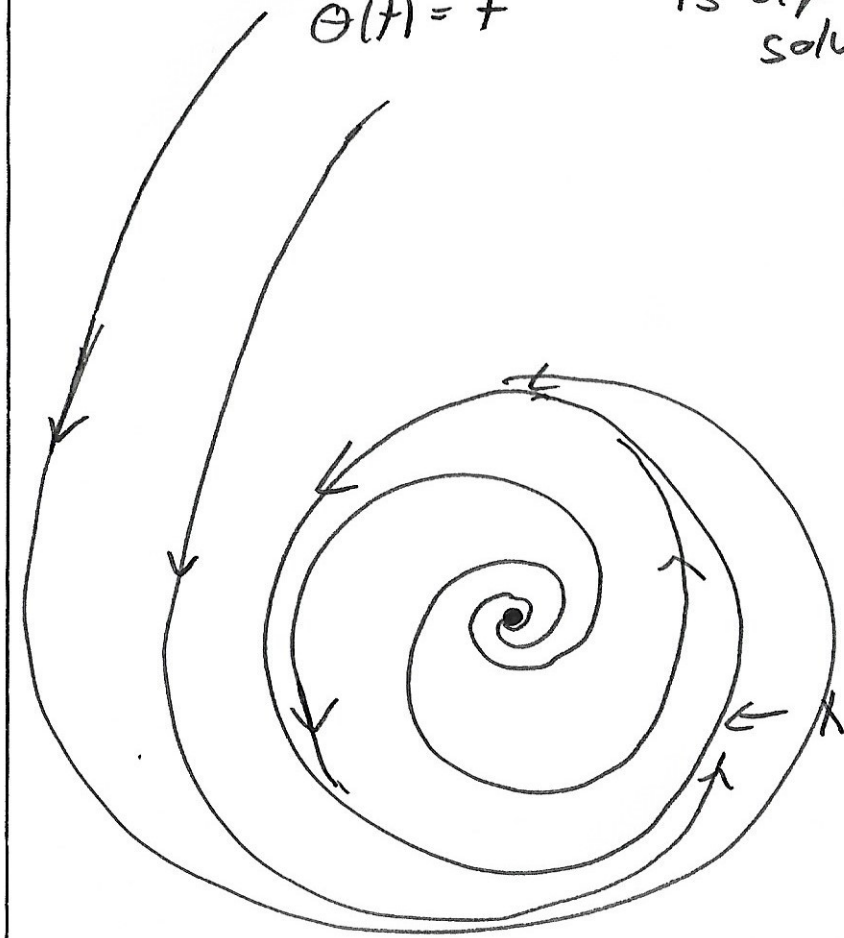
$$\rightsquigarrow \quad r(t) = \frac{1}{1 + (\frac{1}{r_0} - 1) e^{-t}}$$

$$r(0) = r_0.$$

Note: • $t \to \infty$ $\quad r(t) \to 1$

• $r(t) = 1 \quad \forall t$

$\theta(t) = t$ $\qquad$ is a periodic solution



$r_0 < 1 \qquad t \to -\infty \quad r(t) \to 0$

$r_0 > 1 \qquad t \downarrow -T_{r_0} \quad r(t) \to \infty$

$\uparrow$ s.t. $(\frac{1}{r_0} - 1) e^{+T_0} = -1$

2.4 Bendixson-Dulac Theorem

Can <u>rule out</u> existence of non-trivial periodic solutions

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not just a critical points.

<u>Theorem 2.1</u> (Bendixson-Dulac)

Consider $\begin{cases} \dot{x} = X(x,y) \\ \dot{y} = Y(x,y) \end{cases}$, $X, Y \in C^1$

<u>If</u> there is a function $\varphi \in C^1$

s.t.

$$\rho = \partial_x(\varphi X) + \partial_y(\varphi Y) > 0$$

in a simply connected region R.

<u>Then</u> $\nexists$ non-trivial closed trajectories in R.

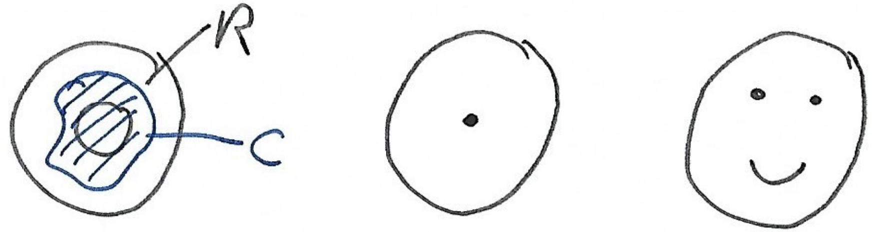
<u>Note:</u>

R simply connected

R C $C \subset R$ closed curve

$\Rightarrow$ D region bounded by C is again in R and --- simply connected.

Non-simply connected:



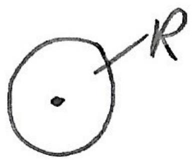
Remark:

suffices to have

$\rho \geq 0$ on R ← simply connected

with $\rho > 0$ on $R \setminus$ "lower dim set"

points   lines



$\rho > 0$ on $D_1 \setminus \{0\}$

on $\mathbb{R}^2 \setminus \{0\}$

on $\mathbb{R}^2 \setminus$ x-axis

good enough.

Need: For all D = 2D subset of R have
$$\iint_D \rho \, dx\,dy > 0.$$

Proof:

param by $(x(t), y(t))$ satisf. $\dot{x} = X$, $\dot{y} = Y$

Suppose C is a closed trajectory in R

Let D = "enclosed disc" = domain $\partial D = C$

Note as C does not have self intersect. and R is simply connected get

$D \subset R$, D simply connected

$\downarrow$ $\rho > 0$ $\qquad$ $\downarrow$ Green's formula applies

$$0 < \iint_D \rho \, dx\,dy = \iint_D \partial_x(\varphi X) + \partial_y(\varphi Y)$$
$$= \oint_{\partial D = C} -\varphi Y dx + \varphi X dy$$
$$= \int_0^T \Big(-\varphi Y \overset{=X}{\dot{x}} + \varphi X \overset{=Y}{\dot{y}}\Big) dt$$
$$= 0$$

↯

Cor. 2.2:

If $\partial_x X + \partial_y Y$ has a fixed sign (zero not allowed) on a simply connected region R then $\nexists$ periodic solution with traj. $\subset R$.

Example:

- Dampered pendulum

$$\dot{x} = y$$
$$\dot{y} = \frac{g}{l} \cdot \sin x - y \cdot k$$

$$\partial_x X + \partial_y Y = -k < 0 \text{ on } \mathbb{R}^2$$

-> no periodic solutions!

- $\ddot{x} + f(x)\dot{x} + x = 0$

$$f(x) > 0 \quad \forall x$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = -f(x) \cdot y - x \end{cases}$$

$$\partial_x X + \partial_y Y = -f(x) < 0$$

-> $\nexists$ periodic solutions.

- $\begin{cases} \dot{x} = y \\ \dot{y} = -x - y + x^2 + y^2 \end{cases}$

$\partial_x X + \partial_y Y = -1 + 2y$. Try $\varphi = \varphi(x)$

$$\partial_x(\varphi X) + \partial_y(\varphi Y) = \varphi_x \overset{=y}{X} + \varphi(1+2y)$$

Want: $\varphi_x + 2\varphi = 0$, $\varphi = -e^{-2x}$

so $\rho = +2e^{-2x} \cdot y - e^{-2x}(1+2y)$
$= +e^{-2x} > 0$ on $\mathbb{R}^2$.