Last time:

$$P(x,y)\,\partial_x z(x,y) + Q(x,y)\,\partial_y z(x,y) = R(x,y,z(x,y)) \qquad \text{(PDE)}$$

+ prescribed data on a curve $(\gamma_1(s), \gamma_2(s))$

Plan: Find solution surface

$$\Sigma = \{(x,y,z(x,y))\}$$

by

- using data to get 1 curve $(\gamma_1(s), \gamma_2(s), \gamma_3(s))$ guaranteed to be in S
- finding special curves = characteristics with the property that if they have one point in Σ then they are fully contained in Σ

→ will get such characteristics $t \mapsto (x(t,s), y(t,s), z(t,s))$ through every point $(\gamma_1(s), \gamma_2(s), \gamma_3(s))$ of data curve

→ parametrisation of Σ via $(s,t) \mapsto (x(s,t), y(s,t), z(s,t))$

3.2 Characteristics

Suppose that $f(x,y)$ solves

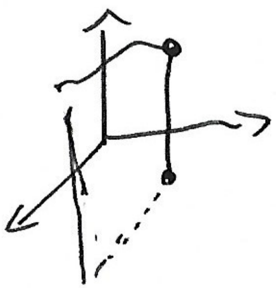
$$P(x,y)\,\partial_x f(x,y) + Q(x,y)\,\partial_y f(x,y) = R(x,y,f(x,y))$$

The normal at $(x,y,f(x,y))$ of

$$\Sigma = \{(x,y,z) \in \mathbb{R}^3 : z = f(x,y)\}$$

is in direction

$$\underline{n} = \begin{pmatrix} -\partial_x f(x,y) \\ -\partial_y f(x,y) \\ 1 \end{pmatrix}$$



$$= \underbrace{\begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix}}_{\text{tangent}} \times \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix} = \text{normal}$$

Note: vector $\underline{\Upsilon}$ is tangential to Σ at $(x,y,f(x,y))$ iff

$$\underline{\Upsilon} \cdot \underline{n} = 0.$$

Special case: If f is a solution of (PDE) then

$$\underline{\Upsilon}(x,y,z) = \begin{pmatrix} P(x,y) \\ Q(x,y) \\ R(x,y,z) \end{pmatrix}$$

satisfies

$$\underline{\Upsilon} \cdot \underline{n} = 0 \quad \text{at points } (x,y,z) \in \Sigma \text{ i.e. } z = f(x,y).$$

<u>Def.</u>: A curve $(x(t), y(t), z(t))$ is a characteristic of (PDE) if

$$(C) \begin{cases} \dot{x}(t) = P(x(t), y(t)) \\ \dot{y}(t) = Q(x(t), y(t)) \\ \dot{z}(t) = R(x(t), y(t), z(t)) \end{cases}$$

(C) = characteristic equations.

Curves $(x(t), y(t), 0)$ are called characteristic projections.

<u>Prop. 3.1</u>:
suppose that
- P, Q Lipschitz-cont. in (x, y)
- R is continuous & satisfies Lip cond w.r.t. z

<u>Then</u>:

(a) $\forall (x_0, y_0) \in \mathbb{R}^2$ $\exists!$ characteristic projection through $(x_0, y_0, 0)$ and char. proj. cannot intersect (but can "meet" at points with $P = 0 = Q$)

$\forall (x_0, y_0, z_0) \in \mathbb{R}^3$ $\exists!$ characteristic

(b) If f is a solution of (PDE), $\Sigma = \{(x, y, z) : z = f(x, y)\}$ and $(x_0, y_0, z_0) \in \Sigma$ then the whole characteristic $(x(t), y(t), z(t))$ is in Σ.

Proof of (a):

- $\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \end{cases}$ plane aut. system

$\to \exists!$ traj. through every (x_0,y_0)

Having thus obtained $(x(t), y(t))$ we can now view

$$\dot{z}(t) = R(\underbrace{x(t), y(t)}_{\text{known}}, z(t))$$

as a ODE for z

$$\dot{z}(t) = \tilde{r}(t, z(t))$$

Picard tells me that $\exists!$ solution as
- $\tilde{r}$ is continuous since R and x, y are cont.
- Lip cond in z.

Proof of (b):

To show: If $(x(t), y(t), z(t))$ satisfies (C) and

$$(x_0, y_0, z_0) = (x, y, z)(0) \in \Sigma$$

then $(x, y, z)(t) \in \Sigma$

i.e. $f(x_0, y_0) = z_0$ + (C) implies that

$$w(t) = f(x(t), y(t)) - z(t) = 0 \quad \forall t$$

Note: $\dot{w}(t) = \partial_x f(x(t), y(t)) \cdot \dot{x}(t) + \partial_y f(x(t), y(t)) \cdot \dot{y}(t) - \dot{z}(t)$ // $R(x,y,f(x,y))$

$$= (\partial_x f)(x,y) \cdot P(x,y) + (\partial_y f)(x,y) Q(x,\ldots$$

$$- R(x, y, z)$$

$$|\mathring{w}(t)| = |R(x,y,f(x,y)) - R(x,y,z)|$$

$$\underset{\substack{\uparrow \\ \text{Lip} \\ \text{cond}}}{\leq} L \cdot |\underbrace{f((x,y))(t)) - z(t)}_{= w(t)}|$$

so $|\mathring{w}(t)| \leq L \cdot |w(t)|$

$w(0) = 0$

Therefore

$$|w(t)| \leq \overset{\overset{0}{\shortparallel}}{w(0)} + \left|\int_0^t \underbrace{|\mathring{w}(\tilde{t})|}_{\leq L \cdot |w(\tilde{t})|} d\tilde{t}\right|$$

$$\leq 0 + L \left|\int_0^t |w(\tilde{t})| d\tilde{t}\right|$$

Gronwall

$$|w(t)| \leq 0 \cdot e^{L|t|} = 0 \quad ▨$$

Ex.:

(b) $y \partial_x z + \partial_y z = z$

(c) $\begin{cases} \dot{x} = y \\ \dot{y} = 1 \\ \dot{z} = z \end{cases}$ $\to y(t) = B + t$

$\downarrow$

$x(t) = Bt + \frac{1}{2}t^2 + A$

$z(t) = C \cdot e^t$

So characteristics are

$(A + Bt + \frac{1}{2}t^2, \ B+t, \ Ce^t)$

$t \in \mathbb{R}$

(a) $x \partial_x z + y \partial_y z = z$

(c) $\begin{cases} \dot{x} = x \\ \dot{y} = y \\ \dot{z} = z \end{cases}$

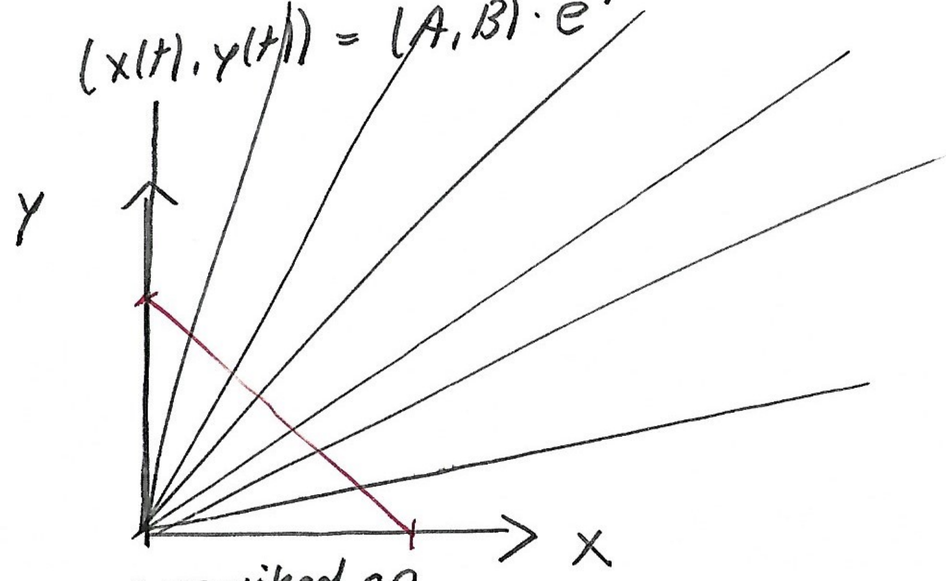
Characteristics = halflines through $(0,0,0)$ in $\mathbb{R}^3$

$= (x(t), y(t), z(t)) = (A,B,C) \cdot e^t$

$t \in \mathbb{R}$

Chav. projections

$(x(t), y(t)) = (A,B) \cdot e^t$



z prescribed on

$y = 1 - x, \quad x \in [0,1]$

e.g. $z(x,y) = x \cdot y$ for such (x,y)

Domain of def = 1st quadrant.