1^{st} order semilinear PDEs

$$P(x,y)\,\partial_x f(x,y) + Q(x,y)\,\partial_y f(x,y) = R(x,y,f(x,y))$$

Last time:

If a characteristic $(x,y,z)(t)$, i.e. solution of (c) $\begin{cases} \dot{x} = P(x,y) \\ \dot{y} = Q(x,y) \\ \dot{z} = R(x,y,z) \end{cases}$

starts on solution surface

$$\Sigma = \{(x,y,z) : z = f(x,y)\}$$

then it remains there.

Today: See how this allows us to solve

"Cauchy problem" $\begin{cases} \text{(PDE)} \\ + \\ \text{"right amount of data"} \end{cases}$

3.3 The Cauchy Problem

A Cauchy problem is a combination of $\begin{cases} \text{(PDE)} \\ + \\ \text{"boundary condition/data and/or initial condition"} \end{cases}$

that in principle we should get a unique solution (on a suitable domain)

Here:

Prescribe z on a curve $\gamma_0 \subset \mathbb{R}^2$ "data curve"

giving us a curve in space = "initial curve" that we ask to be in the solution surface.

Given: $z = g(x,y)$ for (x,y) in γ_0

Parametrising γ_0 as $(\gamma_1(s), \gamma_2(s))$ $s \in I$

$\rightarrow$ get initial curve $(\gamma_1, \gamma_2, \gamma_3)(s)$, $s \in I$

$\gamma_3(s) = g(\gamma_1(s), \gamma_2(s))$.

Method of characteristics

① Parametrise initial curve $(\gamma_1, \gamma_2, \gamma_3)(s)$, $s \in I$.

② For every $s \in I$

solve

$$\begin{cases} \dot{x}(t,s) = P(x,y) \\ \dot{y}(t,s) = Q(x,y) \\ \dot{z}(t,s) = R(x,y,z) \end{cases} \qquad \dot{} = \frac{d}{dt}$$

with
$$x(0,s) = \gamma_1(s)$$
$$y(0,s) = \gamma_2(s)$$
$$z(0,s) = \gamma_3(s)$$

$\rightarrow$ Get ~~its~~ parameter form of solution surface

$$\Sigma = \{ (x,y,z)(t,s) : t \in \dots, s \in I \}$$

③ Try and solve for $z = z(x,y)$

Solve $x = x(t,s)$

$y = y(t,s)$

for s, t

$\rightarrow$ determine

$z = z(x,y)$

$= z(t(x,y), s(x,y))$.

Ex:

(a) $y\,\partial_x z + \partial_y z = z$

$z(x,0) = x \quad , \quad 1 \le x \le 2$

Initial curve

$(s, 0, s) \quad , \quad s \in [1,2]$

characteristics:

$(x, y, z)(t) = (A + Bt + \frac{1}{2}t^2, \; B + t, \; Ce^t)$

so $A = s = C \quad , \quad B = 0$

so $\{ (x,y,z)(t,s) = (s + \frac{1}{2}t^2, \; t, \; se^t)$

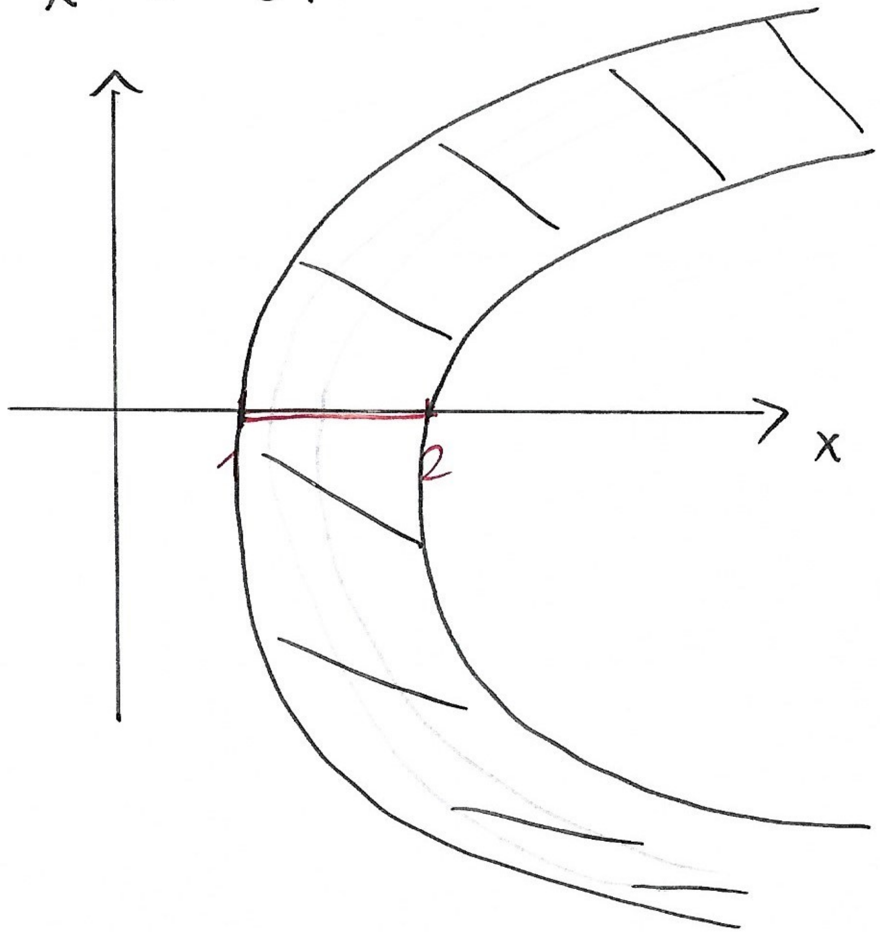
$s \in [1,2], \; t \in \mathbb{R} \}$

Σ

Char. projections:

$x = s + \frac{1}{2}t^2 \quad , \; y = t \quad , \; s \in [1,2]$
$t \in \mathbb{R}$

$\rightsquigarrow x = s + \frac{1}{2}y^2$



Domain of def.

$\Omega = \{ (x,y) : 1 + \frac{1}{2}y^2 \leq x \leq 2 + \frac{1}{2}y^2 \}$

$y = t \quad \rightsquigarrow \quad t = y$

$x = s + \frac{1}{2}t^2 \qquad s = x - \frac{1}{2}y^2$

$z = s e^t$

so $\boxed{\begin{array}{c} z(x,y) = (x - \frac{1}{2}y^2) \cdot e^y \\ (x,y) \in \Omega \end{array}}$