3.5 Domain of definition
& 3.6 Cauchy data

Consider

$$(*)\begin{cases} P\,\partial_x z + Q\,\partial_y z = R \\ z = g(x,y) \text{ on } \hat{\gamma} \subset \mathbb{R}^2 \end{cases}$$

↑ data curve

Parametrising $\hat{\gamma}(s) = (\gamma_1(s), \gamma_2(s))$

can equivalently ask that

initial curve $\gamma \subset \mathbb{R}^3$, $\gamma(s) = (\gamma_1, \gamma_2, \gamma_3)(s)$

is in solution surface,

where $\gamma_3(s) = g(\gamma_1(s), \gamma_2(s))$.

Domain of definition of $(*)$

= set in xy plane s.t.

solution of $(*)$ is

such that the solution $z = f(x,y)$

is <u>uniquely</u> determined by

(PDE) + data.

Domain of def. is swept out by the projections of those characteristics which intersect my initial curve γ

<u>provided</u> we don't have two different such characteristics $\Gamma_1 \neq \Gamma_2$ whose projection to xy-plane is the same curve.

This is guaranteed if the characteristic projections do **NOT** intersect the data curve $\hat{\gamma}$ more than once.

⚠ Characteristics and hence charact. projections might not be defined $\forall t \in \mathbb{R}$ but only for $t \in I_s$

$(x(t,s), y(t,s), z(t,s))$

$s \in$ Interval given by data curve

$t \in I_s$.

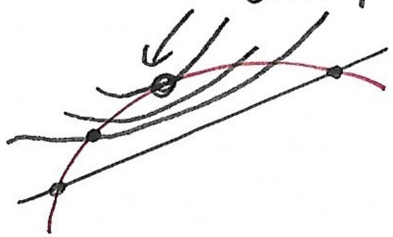
Q: How can we detect if we have a problem with data curve / where we need to split up or restrict the data curve.

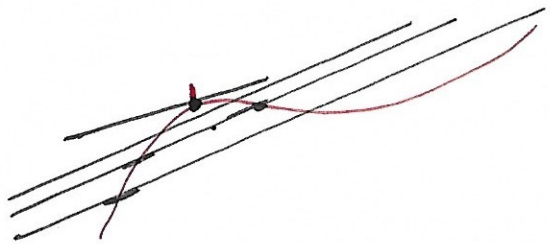
Want:

Condition that guarantees that

- problem NOT overdetermined
- domain of definition is "neighbourhood to the right & left" of data curve.

char. proj. becomes tangential





Need to exclude that characteristic projection is tangential to data curve.

Charact proj. is in direction

$$\begin{pmatrix} \mathring{x}(t,s) \\ \mathring{y}(t,s) \end{pmatrix}$$ where we consider $t=0$ as this coresp. to points on $\hat{\gamma}$

Data curve $s \mapsto (x(0,s), y(0,s))$ is in direction

$$\begin{pmatrix} x'(0,s) \\ y'(0,s) \end{pmatrix} \qquad ' = \frac{\partial}{\partial s}$$

tangential if $\begin{pmatrix} \mathring{x}(0,s) \\ \mathring{y}(0,s) \end{pmatrix}$ and $\begin{pmatrix} x'(0,s) \\ y'(0,s) \end{pmatrix}$ are lin. dependent ie.

if $\det\begin{pmatrix} \mathring{x} & \mathring{y} \\ x' & y' \end{pmatrix} = 0$,

i.e.

$$J(t=0,s) = \det\left(\frac{\partial(x,y)}{\partial(t,s)}\right) = 0$$

$\uparrow$

Jacobian (determinant of Jacobi matrix $= \begin{pmatrix} \partial_t x & \partial_t y \\ \partial_s x & \partial_s y \end{pmatrix}$)

<u>Note</u>: As $\mathring{x}(t,s) = P(\_)$

$\mathring{y} = Q(\_)$

while $(x(0,s), y(0,s)) = (\gamma_1(s), \gamma_2(s))$

we have

$$J(t=0,s) = P(\gamma_1(s), \gamma_2(s)) \cdot \gamma_2'(s) - Q(\gamma_1(s), \gamma_2(s)) \cdot \gamma_1'(s)$$

Def: Data is Cauchy if

$$J(t=0, s) = P(\hat{\gamma}(s)) \cdot \gamma_2'(s) - Q(\hat{\gamma}(s))\, \gamma_1'(s) \neq 0$$

for all s that we consider.

Analytic explanation:

Crucial Q in method:

Can we solve $\begin{array}{l} x = x(s,t) \\ y = y(s,t) \end{array}$

for s and t ?

Inverse function theorem (off syl.)

→ Introd. to mf.

If a C^1 function

$f(x,y) = (f_1(x,y), f_2(x,y))$

is so that

$$\begin{pmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{pmatrix}(p) \text{ is invertible}$$

then f is locally invertible (near p).

$$\ulcorner \quad f \approx \underline{b} + A \cdot \begin{pmatrix} x \\ y \end{pmatrix} \lrcorner$$

If $J(t=0, s) \neq 0$

→ can solve for s and t for

t close to zero

⇝ at least in a nbhd of data curve.

Remark: For P, Q, R analytic
(series expansion)

$\exists!$ solution is guaranteed by Theorem of Cauchy-Kowalewskaya
(Sonya Kowalewskaya (1874))

Back to ex 2:

$$x\,\partial_x z + y \cdot \partial_y z = z$$

Data curve $\hat{\gamma}(s) = (2 - \sqrt{2}\cos s, \sqrt{2}\sin s)$

$$J(0,s) = \det\begin{pmatrix} 2-\sqrt{2}\cos s & \sqrt{2}\cdot\sin s \\ \sqrt{2}\sin s & \sqrt{2}\cdot\cos s \end{pmatrix}$$

$$J(0,s) = 2\sqrt{2}\cos s - 2 \neq 0$$

unless $s = \pi/4$ $\quad (s \in [0, \pi])$

Hence pieces of data curve

$$\{\hat{\gamma}(s) : s \in [0, \pi/4)\}$$

and

$$\{\hat{\gamma}(s) : s \in (\pi/4, \pi]\}$$

are Cauchy.

Remark: If datacurve = part of a char. proj.

then we either have

NO solution

or

∞ many solutions (if initial curve is contained in a characteristic