3.7 Discontinuities in the first derivative

$$P\,\partial_x z + Q\,\partial_y z = R$$

We'll see:

The only curves where solution surface can have discontinuity in first derivatives are characteristics.

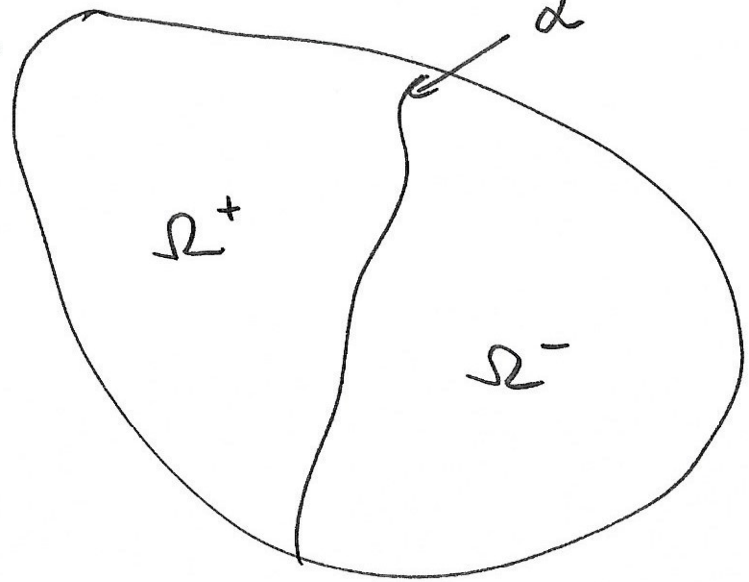
Equivalently:

The only curves in xy-plane along which $\partial_x z, \partial_y z$ can have discontinuity are characteristic projections.

Analytic proof

z solution of PDE

- z continuous, differentiable (away from α)
- $z = z^+$ in Ω^+, $z = z^-$ in Ω^-



Ω = domain of def.

s.t. z^+, z^- satisfy PDE in Ω^+, Ω^-

$z^+ = z^-$ on α

$\begin{pmatrix} \partial_x z^+ \\ \partial_y z^+ \end{pmatrix} \neq \begin{pmatrix} \partial_x z^- \\ \partial_y z^- \end{pmatrix}$ on α

As $z^\pm$ satisfy PDE we have

$$P(x,y)\,\partial_x z^+ + Q(x,y)\,\partial_y z^+ = R(x,y,z^+(\_))$$

$$P(x,y)\,\partial_x z^- + Q(x,y)\,\partial_y z^- = R(x,y,z^-(\_))$$

<u>On α</u> $z^+ = z^-$

$$\to P(x,y)\cdot[\partial_x z]_-^+ + Q(x,y)\cdot[\partial_y z]_-^+ = 0$$

$$[\partial_x z]_-^+ = \partial_x z^+ - \partial_x z^-$$

i.e. $\begin{pmatrix}[\partial_x z]_-^+\\ [\partial_y z]_-^+\end{pmatrix} \perp \begin{pmatrix}P(x,y)\\ Q(x,y)\end{pmatrix}$ on α

direction of charact. projection!

As $z^+ = z^-$ on α

$\to$ parametrise $\alpha(s) = (\alpha_1(s), \alpha_2(s))$

Have

$$0 = \frac{d}{ds}\Big(z^+(\alpha_1(s),\alpha_2(s)) - z^-(\alpha_1(s),\alpha_2(s))\Big)$$

$$= (\partial_x z^+)(\alpha(s))\cdot\alpha_1'(s) - (\partial_x z^-)(\_)\cdot\alpha_1'(s)$$
$$(\partial_y z^+)(\alpha(s))\cdot\alpha_2'(s) - (\partial_y z^-)(\_)\cdot\alpha_2'(s)$$

$$= \begin{pmatrix}[\partial_x z]_-^+\\ [\partial_y z]_-^+\end{pmatrix}\cdot\begin{pmatrix}\alpha_1'(s)\\ \alpha_2'(s)\end{pmatrix}$$

so $\begin{pmatrix}[\partial_x z]_-^+\\ [\partial_y z]_-^+\end{pmatrix} \perp \alpha'(s)$

so $\alpha'(s) \parallel \begin{pmatrix}P\\ Q\end{pmatrix} = \begin{pmatrix}\dot x\\ \dot y\end{pmatrix}$ char. proj.

$\longrightarrow$ α = char. projection.

Ex: $\partial_y z = 0$

$$z(x,0) = \begin{cases} 0 & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\Gamma \quad z = f(x) = \begin{cases} 0 & x \geq 0 \\ -x & x < 0 \end{cases} \quad t$$

Charach: $\begin{pmatrix} A \\ B+t \\ C \end{pmatrix}$