Chapter 4:

2^{nd} order semilinear PDEs

$$a(x,y)\,u_{xx}(x,y) + 2b(x,y)\,u_{xy}(x,y) + c(x,y)\,u_{yy}(x,y) = f(x,y,u(x,y),u_x(x,y),u_y(x,y)) \qquad \text{(PDE)}$$

4.1 Classification and transformation into normal form

<u>Goal:</u> Find "right" coordinates (φ,ψ) in which the PDE is in "normal form"

we'll see that:

If $ac-b^2>0$ "elliptic" $(\Delta u = u_{xx}+u_{yy}=0)$

then get

$$u_{\varphi\varphi}+u_{\psi\psi} = \text{-----}$$

$F(\varphi,\psi,u,u_\varphi,u_\psi)$

If $ac-b^2<0$ "hyperbolic" (wave eq. $u_{tt}=c^2u_{xx}$)

then get

$$u_{\varphi\psi} = \text{-----}$$

If $ac-b^2=0$ "parabolic" $(u_t=u_{xx})$

get $u_{\varphi\varphi} = \text{-----}$

Want change of coordinates

$$(x,y) \longrightarrow (\varphi,\psi)(x,y)$$

to be (locally) invertible so ask that Jacobian

$$J(x,y) = \det\left(\frac{\partial(\varphi,\psi)}{\partial(x,y)}\right) = \det\begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \neq 0$$

$\forall (x,y)$ we consider.

Use "standard abuse of notion" of writing u both for unknown function in (x,y) variables and in the (φ,ψ) variables.

[$u = u(x,y)$

$\cancel{\tilde{u}}(\varphi,\psi)$ s.t. $\cancel{\tilde{u}}(\varphi(x,y),\psi(x,y)) = u(x,y)$]

Need to transform our PDE into our new coordinates.

→ express all derivatives $u_x, u_y, u_{xx} \ldots$ in terms of $u_\varphi, u_\psi, \ldots, \varphi_x, \varphi_y, \ldots$

⇝ get

$$A(\varphi,\psi)\,u_{\varphi\varphi} + 2B(\varphi,\psi)\,u_{\varphi\psi} + C(\varphi,\psi)\,u_{\psi\psi} = F(\varphi,\psi,u,u_\varphi,u_\psi)$$

$u(x,y) = u(\varphi(x,y), \psi(x,y))$

so $\quad u_x = u_\varphi \varphi_x + u_\psi \psi_x$

$\quad u_y = u_\varphi \varphi_y + u_\psi \psi_y$

$$u_{xx} = u_{\varphi\varphi}(\varphi_x)^2 + \overbrace{(u_{\varphi\psi} + u_{\psi\varphi})}^{=2u_{\varphi\psi} \text{ if } u\ C^2} \varphi_x \psi_x$$
$$+ u_{\psi\psi}(\psi_x)^2$$
$$+ u_\varphi \varphi_{xx} + u_\psi \psi_{xx}$$

$$u_{yy} = u_{\varphi\varphi}(\varphi_y)^2 + 2u_{\varphi\psi}\varphi_y\psi_y$$
$$+ u_{\psi\psi}(\psi_y)^2$$
$$+ u_\varphi \varphi_{yy} + u_\psi \psi_{yy}$$

$$u_{xy} = u_{\varphi\varphi}\varphi_x\varphi_y + u_{\varphi\psi}\cdot(\varphi_x\psi_y + \varphi_y\psi_x)$$
$$+ u_{\psi\psi}\psi_x\psi_y$$
$$+ u_\varphi \varphi_{xy} + u_\psi \psi_{xy}$$

so PDE transforms into

$$A(\varphi,\psi)u_{\varphi\varphi} + 2B(\varphi,\psi)u_{\varphi\psi}$$
$$+ C(\varphi,\psi)u_{\psi\psi} = F(\varphi,\psi,u,u_\varphi,u_\psi)$$

where

$$A(\varphi,\psi) = a\cdot\varphi_x^2 + 2b\varphi_x\varphi_y + c\varphi_y^2$$

$$B(\varphi,\psi) = a\varphi_x\psi_x + b(\varphi_x\psi_y + \varphi_y\psi_x)$$
$$+ c\varphi_y\psi_y$$

$$C(\varphi,\psi) = a\psi_x^2 + 2b\psi_x\psi_y + c\psi_y^2$$

(all evaluated at $(x,y) = (x,y)(\varphi,\psi)$)

while F consists of terms coming from

- $f(x(\varphi,\psi), y(\varphi,\psi), u, u_x, u_y)$

  $\qquad\qquad\qquad\qquad\qquad$ " $\quad$ "

  $\qquad\qquad\qquad\qquad\qquad$ ... $\quad$ ...
- lower order terms appearing in e.g. $a u_{xx}$ such as $a(u_\varphi\varphi_{xx} + u_\psi\psi_{xx})$

Note coefficients of principal part transform as

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} = \begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{pmatrix}$$

$\det = \partial(x,y) \neq 0$

so

$$\det \begin{pmatrix} A & B \\ B & C \end{pmatrix} = \partial(x,y)^2 \cdot \det \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

so sign of $ac - b^2$ is invariant under <u>any</u> change of coordinates.

<u>Def</u>: We say that the (PDE) is

- elliptic if $ac - b^2 > 0$
- hyperbolic if $ac - b^2 < 0$
- parabolic if $ac - b^2 = 0$.

⚠ As a, b, c are allowed to depend on (x,y) can have PDEs which have a different type in different parts of xy plane.

$u_{xx} + x \cdot u_{yy} = 0$

$x > 0$ elliptic

$x < 0$ hyperbolic

$x = 0$ parabolic.