Last time

$$a w_{xx} + 2b w_{xy} + c w_{yy}$$

$$= f(x, y, w, w_x, w_y)$$

transforms under

$$(x,y) \to (\varphi, \psi)(x,y)$$

into

$$A w_{\varphi\varphi} + 2B w_{\varphi\psi} + C w_{\psi\psi}$$

$$= F(\varphi, \psi, w, w_\varphi, w_\psi)$$

where

$$A = a(\varphi_x)^2 + 2b\,\varphi_x \varphi_y + c(\varphi_y)^2$$

$$B = a\,\varphi_x \psi_x + b(\varphi_x \psi_y + \varphi_y \psi_x) + c\,\varphi_y \psi_y$$

$$C = a(\psi_x)^2 + 2b\,\psi_x \psi_y + c(\psi_y)^2$$

Today:

Transforming into normal form in

Case 1: Hyperbolic PDEs

i.e. $ac - b^2 < 0$.

Note: Can assume that $a \neq 0$

because: If $a = 0$ the either also

$c = 0 \to$ already have $2b \cdot w_{xy} = f \rightsquigarrow w_{xy} = \frac{f}{2b}$ normal form

or $c \neq 0 \to$ switch x, y variables

Want:

To choose $\varphi = \varphi(x,y)$ s.t.

$$A = a(\varphi_x)^2 + 2b\,\varphi_x \varphi_y + c(\varphi_y)^2 = 0 \qquad (*)$$

Try to understand what this condition means for the levelsets of φ ie. for curves in xy plane on which φ is constant (= "characteristics")

So consider

$$\{(x,y) : \varphi(x,y) = c\}$$

Suppose that this is the graph of a function

$y = y_c(x)$ ie. that we have

$$\{(x,y) : \varphi(x,y) = c\} = \{(x, y_c(x))\}.$$

Note: As $\varphi(x, y_c(x)) = c$ can differentiate to get

$$\varphi_x + \varphi_y \cdot y_c'(x) = 0$$

ie. $\varphi_x = -\varphi_y\, y_c'(x)$.

Back to $(*)$ get

$$0 = (\varphi_y)^2 \cdot \left[a \cdot (y_c')^2 - 2b\, y_c' + c\right]$$

Because $J \neq 0$ can't have $\varphi_y = 0$

so get $\boxed{a(y')^2 - 2b\,y' + c = 0}$

at corresp. point $(x, y(x))$

In order to transform a hyperbolic PDE:

(1) Determine solutions $\lambda_1(x,y)$
$\lambda_2(x,y)$
of the characteristic equation

$$a(x,y)\lambda^2(x,y) - 2b(x,y)\cdot\lambda(x,y) + c(x,y) = 0.$$

Note as $ac - b^2 < 0$ we get 2 distinct real solutions.

(2) Determine the general solution $y_c(x)$ of
$$y_c'(x) = \lambda_1(x, y_c(x))$$

and define $\varphi = \varphi(x,y)$ so that
$$\{(x,y) : \varphi(x,y) = c\} = \{(x, y_c(x))\}$$
(solve for c !)

Same thing with λ_2 gives second new variable ψ

→ Get $(\varphi,\psi)(x,y)$ "characteristic coordinates"

so that
$$A(\varphi,\psi) = 0 \text{ and } C(\varphi,\psi) = 0$$

(3) Transform your PDE into normal form

$$2B(\varphi,\psi)u_{\varphi\psi} = F(\_)$$

$$\boxed{u_{\varphi\psi} = \tilde{F}(\_)} = \frac{F(\_)}{2B}$$

$(B \neq 0$ as $B^2 < AC = 0)$

Example 1 (not in printed lecture notes)

$$x w_{xx} + x^2 w_{xy} = w_x + x^3$$

— correction due to sign error

General solution?

$$a = x, \quad b = \tfrac{1}{2}x^2, \quad c = 0$$

$$ac - b^2 = -\tfrac{1}{4}x^4 < 0$$

$\to$ hyperbolic (if $x \neq 0$)

Char. equation:

$$x \cdot \lambda^2(x,y) - x^2 \cdot \lambda(x,y) = 0$$

so $\lambda_1(x,y) = 0, \quad \lambda_2(x,y) = x$

$$y'(x) = \lambda_1(x, y(x)) = 0$$

$$\to \quad y_c(x) = c$$

so choose

$$\varphi(x,y) = y \qquad \to \varphi_x = 0, \ \varphi_y = 1$$

$$y'(x) = \lambda_2(x, y(x)) = x$$

$$y(x) = \tfrac{1}{2}x^2 + c$$

so choose $\psi = y - \tfrac{1}{2}x^2$

$$\psi_x = -x, \quad \psi_y = 1$$

$$w_x = w_\varphi \overset{=0}{\varphi_x} + w_\psi \psi_x = -x \cdot w_\psi$$

$$w_{xx} = -x \cdot w_{\psi\varphi} \overset{=0}{\varphi_x} + (-x)^2 w_{\psi\psi} - w_\psi$$

$$= x^2 w_{\psi\psi} - w_\psi$$

$$w_{xy} = -x \cdot w_{\psi\varphi} - x w_{\psi\psi}$$

so back to PDE

$$0 = x w_{xx} + x^2 w_{xy} - w_x + x^3$$

$$0 = x^3 \cancel{w_{\psi\psi}} \cancel{- x w_\psi}$$

$$- x^3 w_{\psi\varphi} \cancel{- x^3 w_{\psi\psi}}$$

$$\cancel{+ x w_\psi} + x^3$$

so

$$x^3 w_{\psi\varphi} = x^3$$

i.e. $\boxed{w_{\psi\varphi} = 1}$ normal form

As $\partial_\varphi (w_\psi) = 1$

get $w_\psi = \varphi + \tilde{f}(\psi)$

$$w(\varphi, \psi) = \varphi\psi + f(\psi) + g(\varphi)$$

so

$$w(x,y) = y \cdot (y - \tfrac{1}{2}x^2) + f(y - \tfrac{1}{2}x^2) + g(y)$$

Ex2: Wave equation:

$$w_{xx} - w_{yy} = 0$$

so $ac - b^2 = -1 < 0$ hyperbolic

Char. eq.: $\lambda^2 - 1 = 0 \rightarrow \lambda = \pm 1$

so

$y'(x) = 1 \rightarrow y(x) = x + c$

$$\boxed{\varphi(x,y) = x - y}$$

$y'(x) = -1 \rightarrow y(x) = -x + c$

$$\boxed{\psi(x,y) = x + y}$$

Transformed PDE

$$\boxed{w_{\varphi\psi} = 0}$$

$\partial_\psi (w_\varphi) = 0$

$\rightarrow w_\varphi(\varphi,\psi) = \tilde{f}(\varphi)$

$w(\varphi,\psi) = f(\varphi) + g(\psi)$

General solution of wave equation

$$w(x,y) = f(x-y) + g(x+y).$$

Impose data:

E.g. $w(x,0) = x$, $x \in [0,1]$

$w_y(x,0) = 0$, $x \in [0,1]$.

So get on $(s,0)$, $s \in [0,1]$

$$\begin{cases} f(s) + g(s) = s \\ w_y(s,0) = f'(s)\cdot(-1) + g'(s) = 0 \end{cases}$$

$$\boxed{s \in [0,1]}$$

For $s \in [0,1]$:

$$\begin{cases} f'(s) + g'(s) = 1 \\ f'(s) = g'(s) \end{cases}$$

so $f'(s) = g'(s) = \frac{1}{2}$

so $f(s) = \frac{1}{2}s + c$, $f(\varphi), g(\psi)$ determined
$g(s) = \frac{1}{2}s - c$ for $\varphi \in [0,1]$, $\psi \in [0,1]$

so solution is

$$\boxed{\begin{aligned} w(x,y) &= \tfrac{1}{2}(x-y) + \tfrac{1}{2}(x+y) \\ &\text{for } (x,y) \in \Omega \end{aligned}}$$

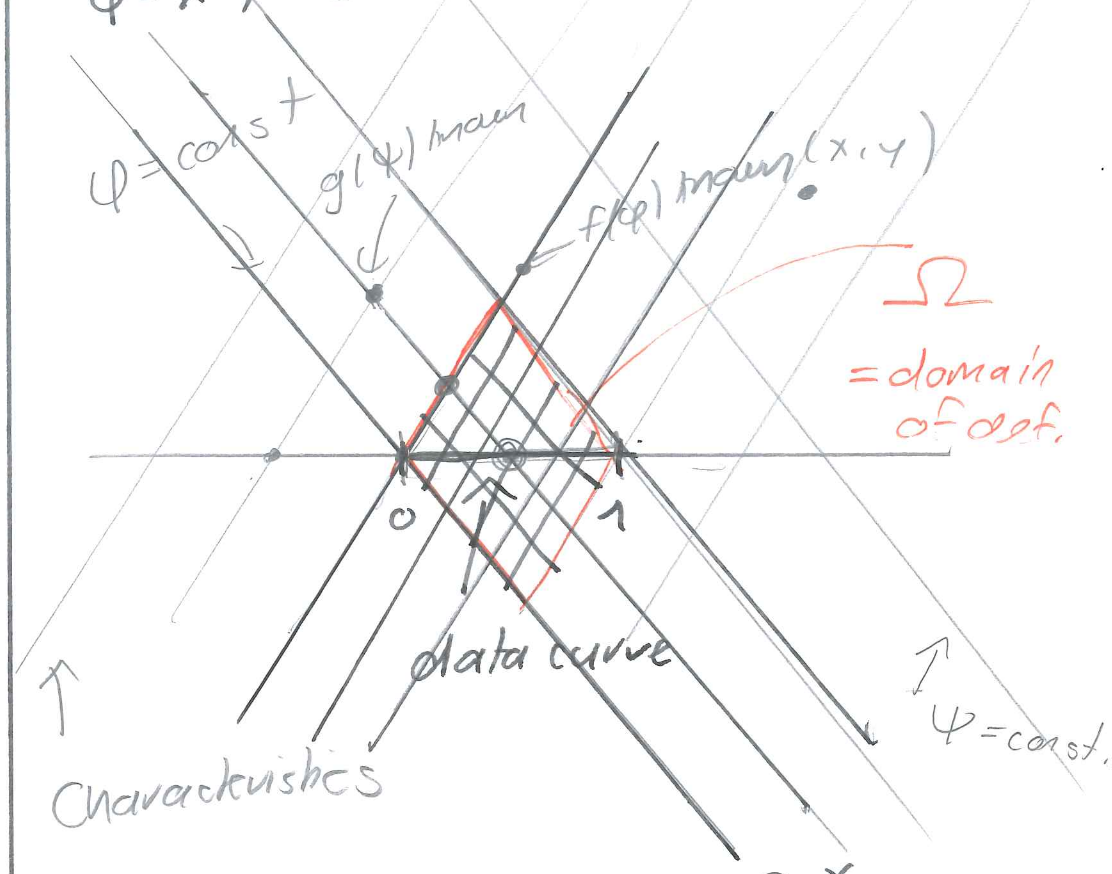
Ω = domain of definition

$= \{(x,y) : \varphi(x,y) \in [0,1],\ \psi(x,y) \in [0,1]\}$

$= \{(x,y) : 0 \le x-y \le 1$ and $0 \le x+y \le 1\}$.

Graphically:

$\varphi = x-y = c$ corresp. to $y = x-c$



$\psi = x+y = c \;\rightarrow\; y = c - x$