Part A Linear Algebra MT 2020, Sheet 1 of 4 [1]

1. Show that the vector space of polynomials $\mathbb{R}[x]$ is isomorphic to a proper subspace of itself.

2. (Harder) Show that the space of functions $f : \mathbb{N} \to \mathbb{R}$ does not have a countable basis.

3. Let $\mathbb{F}$ be a field and $f(x)$ be an irreducible polynomial in $\mathbb{F}[x]$. Show that the set of polynomials modulo $f(x)$ form a field.

4. (a) Show that the set $M_n(R)$ of $(n \times n)$-matrices with entries in a ring R is a ring with the usual matrix addition and multiplication.

   (b) Show that the canonical surjection $R \to R/I$ induces a surjective ring homomorphism $M_n(R) \to M_n(R/I)$. What is the kernel?

   (c) Describe, with justification, the ideals of $M_n(R)$ for a ring R with multiplicative unit 1.

5. Prove that a linear transformation $P : V \to V$ of a finite dimensional vector space satisfies $P^2 = P$ if and only if there exists a basis such that the matrix of P with respect to that basis is a block matrix
$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$
   Hence determine the minimal and characteristic polynomials of P.

6. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space over a field $\mathbb{F}$ to itself. Prove that T is invertible if and only if x does not divide the minimal polynomial $m_T(x)$.

7. Let $T : V \to V$ be a linear transformation of a finite dimensional vector space over a field $\mathbb{F}$ to itself. Assume that $\{v, Tv, T^2v, \dots\}$ span V for some $v \in V$. Show that

   (i) there exists a k such that $v, Tv, \dots, T^{k-1}v$ are linearly independent and for some $\alpha_i \in \mathbb{F}$
$$T^k v = \alpha_0 v + \alpha_1 Tv + \cdots + \alpha_{k-1} T^{k-1} v;$$
   (ii) the set $\{v, Tv, \dots, T^{k-1}v\}$ forms a basis for V;

   (iii) its minimal polynomial is given by $m_T(x) = x^k - \alpha_{k-1}x^{k-1} - \cdots - \alpha_0$.

   What is the characteristic polynomial $\chi_T(x)$?

8. Suppose U is a subspace of V invariant under a linear transformation $T : V \to V$. Prove that T induces a linear map $\overline{T} : V/U \to V/U$ of quotients given by $\overline{T}(v + U) = T(v) + U$. Show further that when V is finite dimensional, the minimal polynomial of $\overline{T}$ divides the minimal polynomial of T.

[1] Many of the problems on these sheets can be found in Kaye & Wilson.

9. Let $\mathcal{P} = \mathbb{F}[x]$ be the vector space of polynomials over the field $\mathbb{F}$. Determine whether or not $\mathcal{P}/\mathcal{M}$ is finite dimensional when $\mathcal{M}$ is

   (i) the subspace $\mathcal{P}_n$ of polynomial of degree less or equal n;

   (ii) the subspace $\mathcal{E}$ of even polynomials;

   (iii) the subspace $x^n\mathcal{P}$ of all polynomials divisible by x^n.

10. Let $\mathcal{P}$ be as above and $L : \mathcal{P} \to \mathcal{P}$ be given by $L(f(x)) = x^2 f(x)$. Prove that L is linear. In the examples above, determine whether L induces a map of quotients $\overline{L} : \mathcal{P}/\mathcal{M} \to \mathcal{P}/\mathcal{M}$. When it does, choose a convenient basis for the quotient space and find a matrix representation of $\overline{L}$.