Part A Linear Algebra MT 2020, Sheet 2 of 4

1. Find all the invariant subspaces of A viewed as a linear map on $\mathbb{R}^2$ or $\mathbb{R}^3$ when A is

$$\begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \quad \begin{pmatrix} 5 & 1 & -1 \\ 0 & 4 & 0 \\ 1 & 1 & 3 \end{pmatrix}.$$

   Now consider A as a linear map on $\mathbb{C}^2$ or $\mathbb{C}^3$. Find the invariant subspaces of A. Also find invertible matrices P such that $P^{-1}AP$ is upper triangular.

2. Let A be an $n \times n$ matrix over $\mathbb{C}$. Show that trace of A is equal to the sum of the eigenvalues, counting each eigenvalue m-times where m is its algebraic multiplicity. Show that the determinant of A is the product of the eigenvalues, again counting algebraic multiplicity. [*Use the upper triangular form, rather than reproducing the proof in Prelims.*]

3. Calculate the minimal and characteristic polynomials of the following matrices.

$$\begin{pmatrix} 1 & 1 & 0 \\ -9 & -4 & 1 \\ -3 & 3 & 2 \end{pmatrix}, \begin{pmatrix} -2 & -3 & -3 \\ -1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -3 & 6 \\ -1 & 1 & -7 \\ 0 & 1 & -3 \end{pmatrix}$$

4. (a) Find two 2×2 matrices over $\mathbb{R}$ which have the same characteristic polynomial but which are not similar.

   (b) Find two 3×3 matrices over $\mathbb{R}$ which have the same minimal polynomial but which are not similar.

   (c) Find two 4×4 matrices over $\mathbb{R}$ which have the same minimal polynomial and the same characteristic polynomial, but which are not similar.

   (d) *[Optional]* Find two nilpotent matrices over $\mathbb{R}$ which have the same minimal polynomial and the same characteristic polynomial, and which have kernels of the same dimension, but which are not similar.

5. The *Fibonacci numbers* x_n are defined by $x_{n+2} = x_{n+1} + x_n$ and $x_0 = 0, x_1 = 1$. Find a formula for x_n in terms of n. [*Hint: Find a two-by-two matrix A that maps (x_n, x_{n+1}) to (x_{n+1}, x_{n+2}).*]

6. Decide whether or not the matrix $A = \begin{pmatrix} 1 & 6 \\ 3 & 5 \end{pmatrix}$ can be diagonalised over the field

   (i) $\mathbb{R}$;

   (ii) $\mathbb{C}$;

   (iii) $\mathbb{Q}$;

   (iv) any field where $1 + 1 = 0$;

   (v) $\mathbb{F}_7 = \{0, 1, 2, 3, 4, 5, 6\}$ with addition and multiplication modulo 7.

7. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Is A diagonisable over $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{F}_3$?

8. Let V be an n-dimensional complex vector space, and let $T\colon V \to V$ be a linear transformation.

(i) Show that for each i, $\ker T^i \subseteq \ker T^{i+1}$, and deduce that there exists a non-negative integer r such that $\ker T^r = \ker T^{r+1}$. Prove that $\ker T^r = \ker T^{r+j}$ for all $j \geq 1$. Hence, or otherwise, show that $V = \ker T^r \oplus \operatorname{Im} T^r$.

(ii) Suppose that the only eigenvalues of T are 0 and λ, where $\lambda \neq 0$. Let $W := \operatorname{Im} T^r$, where r is as above. Show that $T(W) \subseteq W$, and that the restriction of T to W has λ as its only eigenvalue. Let S denote the restriction of $(T - \lambda I)$ to W. Show that 0 is the only eigenvalue of S. By applying (i) with S, W in place of T, V, show that $S^m = 0$ for some m.

9. Let $T : V \to V$ be a linear transformation and suppose that for some $v \in V$, $T^k(v) = 0$ but $T^{k-1}(v) \neq 0$. Prove that the set $\mathcal{B} = \{T^{k-1}(v), \ldots, T(v), v\}$ is linearly independent, and its span U is T-invariant. Find the matrix of T restricted to U relative to the basis $\mathcal{B}$.

10. Let $T : V \to V$ be linear and V be finite dimensional. Assume $m_T(x) = x^m$. Prove that

$$0 \subsetneqq \ker(T) \subsetneqq \ker(T^2) \subsetneqq \cdots \subsetneqq \ker(T^{m-1}) \subsetneqq \ker(T^m) = V.$$

and that these inclusions are indeed strict.