Part A Linear Algebra MT 2020, Sheet 3 of 4

1. For the matrix
$$A = \begin{pmatrix} 0 & 2 & -1 \\ -2 & 3 & -2 \\ -3 & 2 & -2 \end{pmatrix}$$
compute a base-change matrix P such that $P^{-1}AP$ is in Jordan normal form using the following steps:

   (a.) Compute $\chi_A(x)$. Show that it is of the form $-(x-\lambda_1)(x-\lambda_2)^2$ for some distinct λ_1, λ_2.

   (b.) Find basis vectors u of $\text{Ker}(A-\lambda_1 I)$, v_1 of $\text{Ker}(A-\lambda_2 I)$, and v_1, v_2 of $\text{Ker}(A-\lambda_2 I)^2$.

   (c.) Working from first principles, explain why $(A-\lambda_2 I)v_2$ is a scalar multiple of v_1.

   (d.) Let $w_1 = u$, $w_2 = (A-\lambda_2 I)v_2$, and $w_3 = v_2$. What is the matrix of the linear transformation A with respect to this basis? Write down the base-change matrix P.

2. Write down all possible Jordan normal forms for matrices with characteristic polynomial $(x-\lambda)^5$. In each case, calculate the minimal polynomial and the geometric multiplicity of the eigenvalue λ. Verify that this information determines the Jordan normal form.

3. Solve the following system of equations. $x_{n+1} = 2y_n - z_n$, $y_{n+1} = y_n$, $z_{n+1} = x_n - 2y_n + 2z_n$, $x_0 = y_0 = z_0 = 1$. What is the solution in general for x_0, y_0, z_0 arbitrary?

4. Prove that every square matrix over the complex numbers is similar to its transpose. I.e. prove that given any $(n\times n)$-matrix A there exists an $(n\times n)$-matrix P such that $P^{-1}AP = A^t$ where A^t is the transpose of A.

5. Let $\{e_1, e_2, e_3\}$ be the usual basis $\{(1,0,0)^t, (0,1,0)^t, (0,0,1)^t\}$ of $\mathbb{R}^3$. Express the dual basis to
$$\{(1,0,0)^t, (1,-1,1)^t, (2,-4,7)^t\}$$
in terms of e_1', e_2', e_3'.

6. Let S be a set of vectors in V. Define S^0 to be the set of linear functionals that vanish on S. Prove that $S^0 = \langle S\rangle^0$.

7. Suppose that $T : V \to W$ is a linear map and that V is finite dimensional. Prove that $\text{Im}(T') = (\text{Ker}(T))^0$. *[You may assume that $W = \text{Im}(T) \oplus X$ for some subspace X of W.]*

8. Let U be a subspace of V. Show that the restriction map $f \mapsto f|_U$ defines a linear map of dual spaces $V' \to U'$. Hence prove that there is a natural injection $V'/U^0 \to U'$ which is also surjective when V is finite dimensional.

9. (i) Let V be a finite dimensional vector space over $\mathbb{F}$. For a linear transformation $T : V \to V$ define the trace $\text{tr}(T)$ to be the trace of the matrix representing T with respect to some basis $\mathcal{B}$ of V. Show that $\text{tr}(T)$ is well-defined, i.e. show that it is independent of the choice of basis $\mathcal{B}$.

(ii) Let $\mathrm{Hom}(V, V)$ be the space of linear maps from V to itself. For $S \in \mathrm{Hom}(V, V)$ define $f_S : \mathrm{Hom}(V, V) \to \mathbb{F}$ by $T \mapsto \mathrm{tr}(S \circ T)$. Show that f_S is a functional on $\mathrm{Hom}(V, V)$ and that $S \mapsto f_S$ defines a linear isomorphism of $\mathrm{Hom}(V, V)$ to its dual that does not depend on a choice of basis.

10. Let V be finite dimensional. A *hyperplane* in V is defined as the kernel of a linear functional. Show that every subspace of V is the intersection of hyperplanes.