Part A Linear Algebra MT 2020, Sheet 4 of 4

1. Use the Gram-Schmidt process to obtain an orthogonal basis for V, the vector space of polynomials of degree less or equal to two with inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ and basis $\{f_0, f_1, f_2\}$ where $f_0(x) = 1, f_1(x) = x, f_2(x) = x^2$.

2. Let $V = \mathbb{C}^n$. Prove that any sesqui-linear form on V is of the form

$$\langle v, w\rangle = \bar{v}^t Aw$$

for some $n \times n$ matrix A. Furthermore, prove that the form is conjugate symmetric if and only if $A = \bar{A}^t$, and that it is non-degenerate if and only if A is non-singular. When does A define an inner product?

3. Let U and W be subspaces of an inner product space V. Show that

(a) if $U \subseteq W$ then $W^\perp \subseteq U^\perp$;

(b) $(U + W)^\perp = U^\perp \cap W^\perp$;

(c) $U^\perp + W^\perp \subseteq (U \cap W)^\perp$,

and if V is finite dimensional then $U^\perp + W^\perp = (U \cap W)^\perp$.

4. (a) Let V be a finite dimensional real inner product space and $\{e_1, \cdots, e_k\}$ be an orthonormal set of vectors. Let $v \in V$ and write $||v|| = \sqrt{\langle v, v\rangle}$. By considering $||v - \sum_{j=1}^k \langle v, e_j\rangle e_j||^2$, or otherwise, prove the Bessel inequality

$$\sum_{j=1}^k |\langle v, e_j\rangle|^2 \leq ||v||^2.$$

(b) Prove that for elements v, w in an real inner product space V the following inequality holds:

$$|\langle v, w\rangle| \leq \|v\|\|w\|.$$

When does equality hold?

5. Let V be the set of all real sequences (a_n) such that $\Sigma_{n=1}^\infty a_n^2$ converges.

(a) Prove that V is a vector space under component-wise addition and scalar multiplication.

(b) Define a suitable inner product on V and prove that it is an inner product.

(c) Deduce that for all (a_n) and (b_n) in V

$$(\Sigma_{n=1}^\infty (a_n + b_n)^2)^{\frac{1}{2}} \leq (\Sigma_{n=1}^\infty a_n^2)^{\frac{1}{2}} + (\Sigma_{n=1}^\infty b_n^2)^{\frac{1}{2}}.$$

(d) Let U be the subspace of all finite sequences. Show that $U^\perp = \{0\}$, and deduce that $(U^\perp)^\perp = V$ and not equal to U.

6. Let V be an inner product space and $v \in V$.

(a) Suppose T is self-adjoint. Show that $T^2(v) = 0$ implies $T(v) = 0$, and hence $T^n(v) = 0$ for some $n > 0$ implies $T(v) = 0$.

(b) Suppose S and T are both self-adjoint. Show that ST is self-adjoint if and only if S and T commute, i.e. $ST = TS$.

7. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $T((x, y)) = (2ix + y, x)$.

(a) Write down the matrix A of T with respect to the usual basis of $\mathbb{C}^2$.

(b) Is A symmetric? Is A conjugate symmetric?

(c) What are the eigenvectors of A? Is A diagonalisable?

8. Let V be a real inner product space of dimension n and let Q be a self-adjoint linear transformation from V to V.

(a) Suppose that Q is also non-singular. Show that Q^{-1} is non-singular and self-adjoint.

(b) Suppose Q is positive-definite, that is $\langle Q(v), v\rangle$ is positive for all non-zero $v \in V$. Show that the eigenvalues of Q are positive. Deduce that there exists a positive self-adjoint linear transformation S from V to V such that $S^2 = Q$.

(c) Now let P be a self-adjoint linear transformation from V to V. Show that $S^{-1}PS^{-1}$ is self-adjoint. Deduce, or prove otherwise, that there exist scalars $\lambda_1, \dots, \lambda_n$ and linearly independent vectors $e_1, \dots, e_n$ in V such that, for $i, j = 1, 2, \dots, n$:

$$\begin{aligned}&(i)\ Pe_i = \lambda_i Qe_i\\ &(ii)\ \langle Pe_i, e_j\rangle = \lambda_i \delta_{ij}\\ &(iii)\ \langle Qe_i, e_j\rangle = \delta_{ij}\end{aligned}$$

9. Let T be a linear transformation of a finite dimensional complex inner product space V. Show that T^*T is self-adjoint and has only real, non-negative eigenvalues. Let λ be the minimum and μ be the maximum of all eigenvalues. Show that for $v \in V$

$$\lambda^{\frac{1}{2}}||v|| \leq ||T(v)|| \leq \mu^{\frac{1}{2}}||v||.$$

10. (a) Show that the unitary matrices $U(n)$ form a group and that the determinant is a group homomorphism from $U(n)$ onto S^1, the mutiplicative group of complex numbers of length 1. Show that $U(n)$ is not isomorphic to $SU(n) \times S^1$ as a group.

(b) Show that the elements of the group $SU(2)$ are of the form

$$\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}, \qquad \alpha\bar{\alpha} + \beta\bar{\beta} = 1.$$

Deduce that $SU(2)$ can be identified with the 3-sphere S^3, i.e. the elements of length 1 in $\mathbb{C}^2 = \mathbb{R}^4$.

(c) Let $T : V \to V$ be an orthogonal linear transformation of a real inner product space of dimension 3. Show that there is an orthonormal basis B such that ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ is block diagonal with blocks ± 1 and R_θ where R_θ is a rotation by θ.