## Part A Fluid Dynamics and Waves: Sheet 1 HT16

1. Starting from conservation of mass and momentum for a material volume moving in an inviscid fluid, use Reynolds' Transport Theorem to derive the equations

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0, \qquad \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{g}.$$

2. Define the *particle paths* and *streamlines* for a velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$. When do these coincide? Show that a quantity $f(\boldsymbol{x}, t)$ is preserved following the flow if

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} f = 0.$$

Show also that $f$ is constant along streamlines if $\boldsymbol{u} \cdot \boldsymbol{\nabla} f = 0$.

3. An incompressible inviscid fluid is rotating under gravity $g$ with constant angular velocity $\Omega$ about the $z$-axis, which is vertical, so that $\boldsymbol{u} = (-\Omega y, \Omega x, 0)$ relative to fixed Cartesian axes. We wish to find the surfaces of constant pressure, and hence surface of a uniformly rotating bucket of water (which will be at atmospheric pressure).

Bernoulli's equation would imply that $p/\rho + |\boldsymbol{u}|^2/2 + gz$ is constant, so the constant pressure surfaces are

$$z = \text{constant} - \frac{\Omega^2}{2g}\left(x^2 + y^2\right).$$

But this suggests that the surface of a rotating bucket of water is at its highest in the middle. What is wrong?

Write down the Euler equations in component form for this velocity field, integrate them to find the pressure $p$, and hence obtain the correct shape for the free surface.

4. A fluid moves in the region $r \geq a$ outside a circular cylinder, with velocity field given by

$$\boldsymbol{u} = \left(\frac{-\Omega y}{x^2 + y^2}, \frac{\Omega x}{x^2 + y^2}, 0\right),$$

where $\Omega$ is constant. Show that the flow is irrotational. Deduce that

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x} = 0,$$

where $C$ is any simple closed curve that does not enclose the cylinder.

Show that all the streamlines of the flow are circular, and calculate $\Gamma$ when $C$ is a streamline. Deduce that $\Gamma = 2\pi\Omega$ whenever $C$ *does* enclose the cylinder.

5. Consider an incompressible fluid, with constant density $\rho$, subject to a conservative body force, so that $\boldsymbol{g} = -\boldsymbol{\nabla}\chi$ for some potential function $\chi$.

   (a) Show that the momentum equation may be written in the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{\nabla} \times \boldsymbol{u}) \times \boldsymbol{u} = \boldsymbol{\nabla} f$$

   for some function $f$ (which you should define). Deduce from this *Archimedes' Principle*, that an oject immersed in a fluid at rest (so $\boldsymbol{u} \equiv \boldsymbol{0}$) experiences a buoyancy force equal to the weight of fluid displaced. Show also that

   (i) $p + \rho\chi + \frac{1}{2}\rho|\boldsymbol{u}|^2$ is constant along a streamline in steady flow; and

   (ii) $p + \rho\chi + \frac{1}{2}\rho|\boldsymbol{u}|^2$ is constant everywhere in steady irrotational flow.

(b) Show that the *vorticity* $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{u}$ satisfies

$$\frac{\mathrm{D}\boldsymbol{\omega}}{\mathrm{D}t} = (\boldsymbol{\omega} \cdot \boldsymbol{\nabla})\boldsymbol{u}.$$

Deduce that, in two-dimensional incompressible flow, $\boldsymbol{\omega}$ is conserved following the flow.

(c) Let $C(t)$ be a simple closed curve that is convected with the fluid. Define the *circulation* $\Gamma$ around $C$ by

$$\Gamma = \oint_C \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{x}.$$

By transforming to Lagrangian variables, or otherwise (*e.g.* Acheson exercise 5.2), show that $\Gamma$ is independent of $t$. Deduce that a flow which is initially *irrotational* (*i.e.* $\boldsymbol{\omega} = \mathbf{0}$ at $t = 0$) remains irrotational for all time.

(d) Assuming the flow is irrotational, define a *velocity potential* $\phi$ such that $\boldsymbol{u} = \boldsymbol{\nabla}\phi$. Deduce *Bernoulli's equation*, namely

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\boldsymbol{\nabla}\phi|^2 + \frac{p}{\rho} + \chi = F(t).$$

Explain carefully how an appropriate definition of $\phi$ allows the function $F(t)$ to be chosen arbitrarily.

6. From the Euler equations for an incompressible fluid of constant density, deduce the *energy equation*

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_D \frac{1}{2}\rho|\boldsymbol{u}|^2 \,\mathrm{d}x\mathrm{d}y\mathrm{d}z = -\iint_{\partial D} \left(p + \rho\chi + \frac{1}{2}\rho|\boldsymbol{u}|^2\right) \boldsymbol{u} \cdot \boldsymbol{n} \,\mathrm{d}S,$$

where $D$ is a fixed region inside the fluid, enclosed by the fixed closed surface $\partial D$ with outward unit normal $\boldsymbol{n}$.

7. For two-dimensional incompressible flow, with velocity $\boldsymbol{u} = \big(u(x,y,t), v(x,y,t), 0\big)$, show that there exists a *streamfunction* $\psi(x,y,t)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Show that $\psi$ is constant along streamlines and that the difference between the values of $\psi$ on any two streamlines is equal to the net flux of fluid between them.

8. Suppose that the pressure $p$ in a compressible fluid is a function of the density $\rho$, so that $p = P(\rho)$.

(a) By linearising around a rest state ($\boldsymbol{u} \equiv \mathbf{0}$) with uniform density $\rho_0$, show that the density perturbation $\rho'(\boldsymbol{x}, t)$ obeys the wave equation

$$\frac{\partial^2 \rho'}{\partial t^{2^2}} = c^2 \boldsymbol{\nabla}^2 \rho',$$

and find an expression for the wave speed $c$.

(b) Show that if the flow is irrotational, each component of the velocity perturbation $\boldsymbol{u}'$ also satisfies the same wave equation. Why would it be reasonable to assume that the flow is irrotational?