Part A Fluid Dynamics and Waves: Sheet 2

1. A line source of strength Q is at $z = a$, and a line sink of the same strength is at $z = -a$, where $a \in \mathbb{R}^+$. Write down the complex potential $w(z)$. Find $\mathrm{d}w/\mathrm{d}z$, locate any stagnation points and sketch the streamlines.

   Now let $a \to 0$ and $Q \to \infty$ while keeping the product aQ fixed. This gives the flow due to a *doublet*. Show that its complex potential is μ/z, where μ is to be found in terms of a and Q. Show that the streamlines are circles through the origin with centres on the y-axis.

2. The velocity field $\boldsymbol{u} = (Q/2\pi r)\boldsymbol{e}_r$, in terms of plane polar coordinates (r, θ), corresponds to a *line source* if $Q > 0$ or a *line sink* if $Q < 0$. Show that it is irrotational and incompressible for $r > 0$. Find the velocity potential and streamfunction, and show that the complex potential is

$$w = \frac{Q}{2\pi} \log z.$$

   Explain why the streamfunction is a multi-valued function of position.

   Fluid occupies the region $x > 0$ and there is a plane rigid boundary at $x = 0$. Find the complex potential for the flow due to a line source at the point $(d, 0)$, where $d > 0$, and show that the pressure at $x = 0$ decreases to a minimum at $|y| = d$ and thereafter increases with y.

3. Incompressible inviscid fluid occupies the region $y > 0$, and there is a rigid plane wall at $y = 0$. There is a uniform flow, speed U, in the positive x-direction, and a line source of strength Q at $(0, a)$, where $a > 0$. Find the complex potential $w(z)$ and calculate $\mathrm{d}w/\mathrm{d}z$. Let $\beta = Q/2\pi Ua$. Show that if $\beta > 1$ there are two stagnation points, both on the wall, while if $\beta < 1$ there is only one, in the fluid, a distance a from the origin. Try to sketch the streamlines in either case.

4. Consider the steady, two-dimensional, irrotational flow of a fluid with constant density ρ past a closed body B. If p and w are the pressure and complex potential of the flow, and $z = x + \mathrm{i}y$, show that the force exerted on B by the fluid is (F_x, F_y), where

$$F_x + \mathrm{i}F_y = \oint_{\partial B} p\,\mathrm{i}\,\mathrm{d}z = -\frac{\mathrm{i}\rho}{2} \oint_{\partial B} \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 \mathrm{d}z.$$

   Explain why the integral on the right-hand side is not amenable to calculation via Cauchy's Theorem as it stands.

   By taking the complex conjugate, or otherwise, deduce *Blasius' Theorem*:

$$F_x - \mathrm{i}F_y = \frac{\mathrm{i}\rho}{2} \oint_{\partial B} \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 \mathrm{d}z.$$

5. State and prove *Milne-Thomsons Circle Theorem*. Inviscid, incompressible fluid occupies the region $x^2 + y^2 > a^2$ outside a rigid circular cylinder of radius a. There is a line source of strength Q at $(b, 0)$, where $b > a$, and there is also a circulatory flow around the cylinder as if due to a line vortex of strength Γ at the origin. Explain why the complex potential is

$$w(z) = \frac{Q}{2\pi} \log(z - b) + \frac{Q}{2\pi} \log\left(\frac{a^2}{z} - b\right) - \frac{\mathrm{i}\Gamma}{2\pi} \log z.$$

Calculate $\mathrm{d}w/\mathrm{d}z$ and use Blasius Theorem to find the force components (F_x, F_y) on the cylinder.

6. A two-dimensional irrotational incompressible flow has streamfunction $\psi = A(x-c)y$, where A and c are constants. A circular cylinder of radius a is introduced, its centre being at the origin. Find the force exerted on the cylinder by the resulting flow.