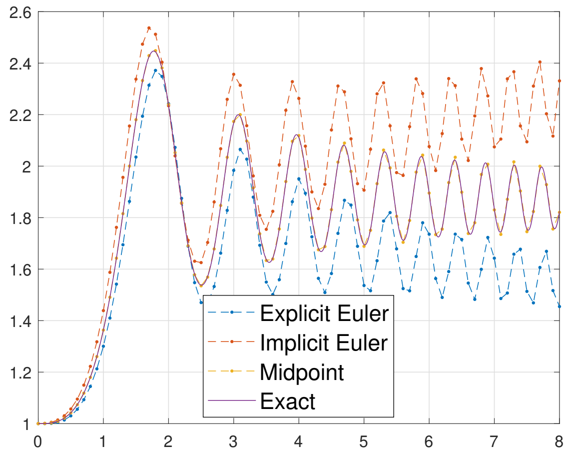
Numerical Analysis

Sheet 4 — HT21

Solving initial value problems

1. Consider the scalar IVP $y' = \sin(x^2)y$, $y(0) = 1$. Compute the approximation of $y(0.1)$ obtained using one step of the (i) explicit Euler method, (ii) implicit Euler method, and (iii) implicit Midpoint rule.

   FYI below is a plot of the approximate solutions along with the exact one.



2. Consider the autonomous ODE $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ and compute the consistency order of the explicit Euler method.

3. Write the formula of the stages $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4$ and express $\mathbf{y}_{n+1}$ in terms of $\mathbf{y}_n, h$ and $\mathbf{k}_i$ for the following Runge-Kutta method

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & 1/6 & 2/6 & 2/6 & 1/6 \end{array} .$$

   Provide an upper bound of its consistency order.

4. Write the Butcher table of the Runge-Kutta method defined by

$$\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{h}{2}\mathbf{f}(x_n, \mathbf{y}_n) + \frac{h}{2}\mathbf{f}(x_n + h, \mathbf{y}_{n+1}) .$$

   and determine its order of convergence.

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5. (a) Derive the formula of the stability function of the explicit Euler, implicit Euler, and the implicit midpoint rules.

   (b) Show that the implicit midpoint rule is A-stable. [*Hint:* You could use the maximum principle for holomorphic functions.]

   (c) Show that the implicit Euler method is L-stable.

6. (a) Write the first and second characteristic polynomials of the explicit Euler, implicit Euler, and implicit trapezium rules.

   (b) Show that these methods are zero-stable.

   (c) Show that the implicit Euler and implicit trapezium rules are A-stable using the definition of stability domain of multistep methods.

7. Let $a, b \in \mathbb{R}$ be some fixed parameters. Show that the multistep methods described by

$$\rho(z) = (z-1)(az+1-a), \quad \sigma(z) = (z-1)^2 b + (z-1)a + (z+1)/2$$

   are of order 2, and show that they are zero-stable if and only if $a \geq 1/2$.

8. (Optional)

   (a) Prove that the stability function of any explicit s-stage Runge-Kutta method is a polynomial of degree at most s.

   [*Hint:* show by induction that the i-th stage $k_i(z)$ is a polynomial in z of degree at most i.]

   (b) Prove that the stability function of any explicit s-stage Runge-Kutta method of order s is exactly $S(z) = \sum_{j=0}^{s} \frac{z^j}{j!}$.

9. (Optional) Show that $hD = -\log(\mathbf{I} - \Delta)(\mathbf{I} - \Delta)E$ and that

$$hD = \left(\Delta - \frac{1}{2}\Delta^2 - \frac{1}{6}\Delta^3 + \dots\right) E,$$

   and write the formulas of the first and the second characteristic polynomials of the 1-step and 2-step methods associated to this series. Are these methods zero-stable?

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