Numerical Analysis

Sheet 3 — HT21

Orthogonal polynomials, best approximation, quadrature

1. For each of the following, say if it defines a norm on $C^1[a,b]$ (the vector space of continuously differentiable functions on $[a,b]$), and if not, why not:

   (i) $\left|\int_a^b f(x)\,\mathrm{d}x\right|$

   (ii) $\max_{x\in[a,b]} |f(x)+f'(x)|$

   (iii) $\max_{x\in[a,b]} [f(x)]^2$

   (iv) $\max_{x\in[a,b]} \{|f(x)|+|f'(x)|\}$

2. Calculate the orthogonal polynomials ϕ_0, ϕ_1, ϕ_2 in the inner product space defined by

$$\langle f,g\rangle = \int_0^2 x f(x) g(x)\,\mathrm{d}x.$$

3. Calculate the best approximation to x^3 on $[0,2]$ from Π_2 in the norm derived from the inner product as above,

$$\int_0^2 x f(x) g(x)\,\mathrm{d}x = \langle f,g\rangle.$$

   [If you like you can use Matlab or Python for solving linear systems]

4. By considering $\|f-(p+\epsilon q)\|^2$ where $\epsilon\in\mathbb{R}$, $q\in\Pi_n$, show that if $p\in\Pi_n$ is a best approximation to f in this norm with associated inner product $\langle\cdot,\cdot\rangle$ then $\langle f-p,q\rangle=0$ for any $q\in\Pi_n$.

5. If $\{\phi_0,\phi_1,\ldots,\phi_n,\ldots\}$ are orthogonal polynomials in $\langle\cdot,\cdot\rangle$ which are normalised to be monic (i.e. have leading coefficient equal to 1) show that $\|\phi_k\|\le\|q\|$ for all monic polynomials $q\in\Pi_k$ which are of exact degree k where $\|\cdot\|$ is the norm derived from the inner product.

6. Let $\mu_j=\int_a^b x^j w(x)\,\mathrm{d}x$ be the jth *moment* of the weight distribution $w(x)$. Show that the linear system of equations

$$\begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} = \begin{bmatrix} \mu_n \\ \mu_{n+1} \\ \vdots \\ \mu_{2n-1} \end{bmatrix}$$

   has as solution the coefficients of a polynomial $x^n-\sum_{j=0}^{n-1} c_j x^j$, which is a member of the family of orthogonal polynomials associated with the weight function w.

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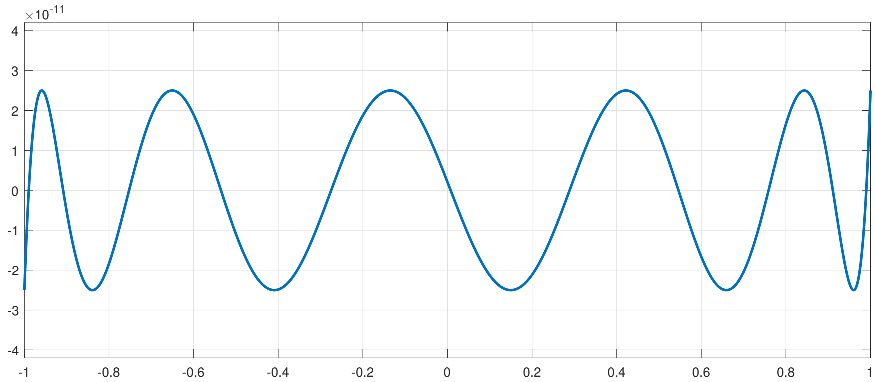
7. Let $p(x) = \sum_{k=0}^{n} c_k\phi_k(x)$ where $\{\phi_k\}_{k=0}^{n}$ are the *orthonormal* Legendre polynomials on $[-1,1]$. (i) What is $\int_{-1}^{1} p(x)dx$? (ii) What is the best degree-k polynomial approximant to p in the L_2-norm? (i.e., minimiser of $\int_{-1}^{1}(p(x)-q_k(x))^2dx$ over $q_k \in \Pi_k$)

8. Let $f : [a,b] \to \mathbb{R}$ be a real continuous function. Consider finding the best degree-k polynomial approximant p_k to f on $[a,b]$ in the L_∞-norm (also known as minimax approximation). The solution is known to have a beautiful "equioscillation" property. For example, below is the error $\exp(x) - p_{10}(x)$ of the degree 10 minimax polynomial approximant to the exponential function on $[-1,1]$.



Make this precise by proving that equioscillation implies optimality: If $f - p_k$ has $k+2$ extrema $(a \le)x_1 < x_2 < \cdots < x_{k+2}(\le b)$ with alternating signs, i.e., $f(x_i) - p_k(x_i) = (-1)^{i+\sigma}\|f-p_k\|_\infty$ where $\sigma = 0$ or 1, then p_k is a minimax polynomial approximant of degree k to f.

Note 1: such computation can be done conveniently using Chebfun as e.g.

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f = chebfun(@(x)exp(x)); p = minimax(f,10); plot(f-p).
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Note 2: The equioscillation condition is in fact necessary and sufficient.

9. *Simpson's Rule* is a quadrature rule based on taking three sample points (endpoints x_0, x_2 and the center x_1), finding the quadratic polynomial interpolant, and integrating it. Show that Simpson's rule applied to $I = \int_{x_0}^{x_2} f(x)dx$ gives the approximation $I \approx \frac{x_1-x_0}{3}[f(x_0)+4f(x_1)+f(x_2)]$.

   Show further that Simpson's rule is exact if f is a cubic polynomial.

10. (Optional:) Let f be a polynomial of degree $2n+1$, expressed as $f(x) = \sum_{i=0}^{2n+1} c_iP_i(x)$, where $\{P_i(x)\}_{i=0}^{2n+1}$ are orthonormal polynomials satisfying $\int_{-1}^{1} P_i(x)P_j(x)dx = \delta_{ij}$ (i.e., scaled Legendre polynomials).

    (a) Explain how to compute c_0 exactly by sampling f at $n+1$ points.

    (b) Explain how to compute c_1 exactly by sampling f at $n+2$ points.

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