Numerical Analysis

Sheet 2 — HT21

SVD, least-squares and eigenvalues

1. (a) A 2×2 Givens rotation is a matrix of the form $J = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$, where $c = \cos\theta, s = \sin\theta$. Verify that J is orthgonal, and show that for any $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, there exists θ such that $J \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} C \\ 0 \end{bmatrix}$. What are the possible values of C?

   (b) A general $n \times n$ Givens rotation $J(i,j)$ is equal to the identity I_n except for the $(i,i), (i,j), (j,i), (j,j)$ entries, which are equal to $c, s, -s, c$ respectively (so they form the same matrix $\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$). Show that if $x \in \Re^n$ then (denoting by $J(i,j)$ a Givens rotation)

$$J(1,n)J(1,n-1)\cdots J(1,3)J(1,2)\,x = \begin{bmatrix} \beta \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

   for some β and further prove that $\beta^2 = x^T x$.

2. Consider the least-squares problem

$$\min_x \|Ax - b\|, \qquad A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^{m\times 1}, m \geq n. \tag{1}$$

   Suppose that $\text{rank}(A) = n$.

   (a) Derive a solution x based on the (full or thin) QR factorisation of A.

   (b) Show that $x = (A^T A)^{-1} A^T b$ is also the solution for (1). (The first QR-based method has better numerical stability and hence preferred)

3. Show that all the eigenvalues of a real symmetric matrix A are real, and eigenvalues of an orthogonal matrix are on the unit circle.

4. What is the SVD of a normal matrix A, with respect to the eigenvalues and eigenvectors? What if A is symmetric? And unitary?

5. If $A \in \mathbb{R}^{n\times n}$ is nonsingular, what is the SVD of A^{-1} in terms of that of A?

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6. Give estimates based on Gershgorin's theorem for the eigenvalues of

$$A = \begin{bmatrix} 9 & 1 & 0 \\ 1 & 4 & \varepsilon \\ 0 & \varepsilon & 1 \end{bmatrix}, \quad |\varepsilon| < 1.$$

   Find a way to establish the tighter bound $|\lambda_3 - 1| \leq \varepsilon^2$ on the smallest eigenvalue of A. (Hint: consider diagonal similarity transformations.)

7. We have seen how the QR algorithm computes the eigenvalue decomposition of symmetric matrices. Using QR, describe an algorithm that computes the SVD of $A \in \mathbb{R}^{m\times n}(m \geq n)$. You can assume that $\text{rank}(A) = n$.

8. The (unshifted) QR algorithm performs: $A_1 = A$, and for $k = 1, 2, \ldots$ form the QR factorisation $A_k = Q_k R_k$, and set $A_{k+1} = R_k Q_k$.

   Let A be a 10×10 symmetric matrix with eigenvalues $1, 2, \ldots, 10$. Which (off-diagonal) elements of A_k converge to 0 the fastest? (Recall the connection between the QR algorithm and power method).

9. (Computational) Implement the QR algorithm in MATLAB or Python, and explore the following. (this exercise should be doable in about 10 lines of code!)
   (a) Verify the above problem in your code. (take $A = Q\Lambda Q^T$, where Q is a randomly generated orthogonal matrix, e.g. `[Q,R]=qr(randn(n));`).
   (b) What will happen if the QR algorithm (without shifts) is applied to an orthogonal matrix A? Explain why this is not a 'counterexample' for the convergence of the QR algorithm. Modify the algorithm so that it computes an eigenvalue decomposition of A.

10. (Optional) Let $A \in \mathbb{C}^{n\times n}$ and define

$$C_{i,j} = \left\{ z \in \mathbb{C} \,\middle|\, |a_{ii} - z||a_{jj} - z| \leq \left(\sum_{\substack{k\neq i \\ k=1}}^{n} |a_{ik}| \right) \left(\sum_{\substack{k\neq j \\ k=1}}^{n} |a_{jk}| \right) \right\}, \quad 1 \leq i < j \leq n.$$

   These are called the *ovals of Cassini.*
   (a) Prove that all eigenvalues of A lie in the union of the ${}_nC_2 = \frac{n(n-1)}{2}$ regions $\cup_{i<j} C_{i,j}$.
   (b) Prove that $C := \cup_{i<j} C_{i,j} \subseteq D$, that is, C is a subset of the union of the Gerschgorin disks $D := \cup_i \{z \in \mathbb{C} |\ |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}|\}$.

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