Numerical Analysis

Sheet 1 — HT21

Lagrange interpolation and LU,QR

1. Construct the Lagrange interpolating polynomial for the data

x	0	1	3
f	3	2	6

.

2. If $p_n \in \Pi_n$ interpolates f at $x_0, x_1, \ldots, x_n$, prove that $p_n + q$ is the Lagrange interpolating polynomial to $f + q$ at $x_0, x_1, \ldots, x_n$ whenever $q \in \Pi_n$.

3. Consider interpolating $1/x$ by $p_n \in \Pi_n$ (i.e. at $n+1$ points) on $[1, 2]$. If $e(x)$ is the error, show that $|e(x)| \leq 1$ for $x \in [1, 2]$ with arbitrarily distributed points, but $|e(x)| \leq 1/2^{(n+1)/2}$ for all $x \in [1, 2]$ if $n + 1$ is even and half of the interpolation points are in $[1, \frac{3}{2}]$ and half in $(\frac{3}{2}, 2]$. In this latter situation, how many points would be needed to guarantee $|e(x)| \leq 10^{-3}$?

4. Show that $\sum_{k=0}^{n} q(x_k) L_{n,k}(x) = q(x)$ whenever $q \in \Pi_n$. (*Optional:* How many ways can you prove this?) Also, show that $\sum_{k=0}^{n} x_k^l L_{n,k}(x) = x^l$ for nonnegative integers $l \leq n$.

5. By performing Gauss Elimination (without pivoting), solve

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \\ 24 \\ 25 \end{bmatrix}.$$

From your calculations, write down an LU factorisation of the matrix A above, and verify that $LU = A$. Then by successive back and forwards substitutions (and without further factorisation) solve $Ax = b_2$ where $b_2 = [4\ 7\ 9\ 2]^T$.

6. What is the determinant of the matrix A in the question above? (Note one of the few algebraic properties of the determinant is that $\det(BC) = \det(B)\det(C)$ and you might also want to consider what is the determinant of a triangular matrix).

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7. Suppose A is a real $n \times n$ matrix with $n \geq 2$ and that the permutation matrix

$$P = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & . & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}.$$

Show that premultiplication of A by P reverses the order of the rows of A.

If $A = LU$ is an LU factorisation of A (without pivoting), what is the structure of PLP? Hence describe how to calculate a factorisation $A = \hat{U}\hat{L}$ where $\hat{U}$ is unit upper triangular and $\hat{L}$ is lower triangular.

8. Suppose that A is a square nonsingular matrix. Prove that the factors Q and R featuring in the QR factorisation of A are unique if the diagonal entries of R are all positive. How many possibilities are there if this restriction is removed?

9. By considering the QR factorisation in which the diagonal entries of R are all positive as in the question above (or otherwise), prove that any orthogonal matrix may be expressed as the product of Householder matrices.

10. Prove that the product of two lower triangular matrices is lower triangular and that the inverse of a non-singular lower triangular matrix is lower triangular. Deduce similar results for upper triangular matrices.

11. (MATLAB/Python exercise) Using a loop and `tic` and `toc` compare the time it takes to do (pivoted) LU and QR factorisations. For example, for random matrices of dimension 2^5 to 2^{10}

```
for k=5:10, A=randn(2^k); tic, [L,U,P]=lu(A); toc,...
tic, [Q,R]=qr(A); toc, end
```

should give some timings. What do you think the computational work is for QR factorisation given that LU is to leading order $\frac{2}{3}n^3$? Note `qr` uses Householder matrices as described in lectures to compute the QR factorisation.

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12. (optional:) Given an LU factorisation of a matrix A, how might one calculate a column of the inverse of A? Estimate the computational work in calculating A^{-1} and hence in solving $Ax = b$ via explicit computation of A^{-1} and multiplication by b.

    Are you now convinced that this is *not* the way to solve linear systems of equations in practice?!

    An even worse technique would be to apply GE separately for each column: what would the computational cost be then?

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