
Numerical Analysis Hilary Term 2021
Lecture 10: Orthogonal Polynomials

Gram–Schmidt orthogonalization procedure: the solution of the normal equations $A\alpha = \varphi$ for best least-squares polynomial approximation would be easy if A were diagonal. Instead of $\{1, x, x^2, \dots, x^n\}$ as a basis for Π_n , suppose we have a basis $\{\phi_0, \phi_1, \dots, \phi_n\}$.

Then $p_n(x) = \sum_{k=0}^n \beta_k \phi_k(x)$, and the normal equations become

$$\int_a^b w(x) \left(f(x) - \sum_{k=0}^n \beta_k \phi_k(x) \right) \phi_i(x) dx = 0 \quad \text{for } i = 0, 1, \dots, n,$$

or equivalently

$$\sum_{k=0}^n \left(\int_a^b w(x) \phi_k(x) \phi_i(x) dx \right) \beta_k = \int_a^b w(x) f(x) \phi_i(x) dx, \quad i = 0, \dots, n, \quad \text{i.e.,}$$

$$A\beta = \varphi, \tag{1}$$

where $\beta = (\beta_0, \beta_1, \dots, \beta_n)^T$, $\varphi = (f_1, f_2, \dots, f_n)^T$ and now

$$a_{i,k} = \int_a^b w(x) \phi_k(x) \phi_i(x) dx \quad \text{and} \quad f_i = \int_a^b w(x) f(x) \phi_i(x) dx.$$

So A is diagonal if

$$\langle \phi_i, \phi_k \rangle = \int_a^b w(x) \phi_i(x) \phi_k(x) dx \quad \begin{cases} = 0 & i \neq k \text{ and} \\ \neq 0 & i = k. \end{cases}$$

We can create such a set of **orthogonal polynomials**

$$\{\phi_0, \phi_1, \dots, \phi_n, \dots\},$$

with $\phi_i \in \Pi_i$ for each i , by the Gram–Schmidt procedure, which is based on the following lemma.

Lemma. Suppose that ϕ_0, \dots, ϕ_k , with $\phi_i \in \Pi_i$ for each i , are orthogonal with respect to the inner product $\langle f, g \rangle = \int_a^b w(x) f(x) g(x) dx$. Then,

$$\phi_{k+1}(x) = x^{k+1} - \sum_{i=0}^k \lambda_i \phi_i(x)$$

satisfies

$$\langle \phi_{k+1}, \phi_j \rangle = \int_a^b w(x) \phi_{k+1}(x) \phi_j(x) dx = 0, \quad j = 0, 1, \dots, k, \quad \text{with}$$

$$\lambda_j = \frac{\langle x^{k+1}, \phi_j \rangle}{\langle \phi_j, \phi_j \rangle}, \quad j = 0, 1, \dots, k.$$

Proof. For any j , $0 \leq j \leq k$,

$$\begin{aligned} \langle \phi_{k+1}, \phi_j \rangle &= \langle x^{k+1}, \phi_j \rangle - \sum_{i=0}^k \lambda_i \langle \phi_i, \phi_j \rangle \\ &= \langle x^{k+1}, \phi_j \rangle - \lambda_j \langle \phi_j, \phi_j \rangle \\ &\quad \text{by the orthogonality of } \phi_i \text{ and } \phi_j, i \neq j, \\ &= 0 \quad \text{by definition of } \lambda_j. \end{aligned} \quad \square$$

- Notes:**
1. The G–S procedure does this successively for $k = 0, 1, \dots, n$.
 2. ϕ_k is always of exact degree k , so $\{\phi_0, \dots, \phi_\ell\}$ is a basis for $\Pi_\ell \forall \ell \geq 0$.
 3. ϕ_k can be normalised to satisfy $\langle \phi_k, \phi_k \rangle = 1$ or to be monic, or ...

Examples: 1. The inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx$

gives orthogonal polynomials called the **Legendre polynomials**,

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = x, \quad \phi_2(x) = x^2 - \frac{1}{3}, \quad \phi_3(x) = x^3 - \frac{3}{5}x, \dots$$

2. The inner product $\langle f, g \rangle = \int_{-1}^1 \frac{f(x)g(x)}{\sqrt{1-x^2}} dx$

gives orthogonal polynomials called the **Chebyshev polynomials**,

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = x, \quad \phi_2(x) = 2x^2 - 1, \quad \phi_3(x) = 4x^3 - 3x, \dots$$

3. The inner product $\langle f, g \rangle = \int_0^\infty e^{-x} f(x)g(x) dx$

gives orthogonal polynomials called the **Laguerre polynomials**,

$$\begin{aligned} \phi_0(x) &\equiv 1, \quad \phi_1(x) = 1 - x, \quad \phi_2(x) = 2 - 4x + x^2, \\ \phi_3(x) &= 6 - 18x + 9x^2 - x^3, \dots \end{aligned}$$

Lemma. Suppose that $\{\phi_0, \phi_1, \dots, \phi_k, \dots\}$ are orthogonal polynomials for a given inner product $\langle \cdot, \cdot \rangle$. Then, $\langle \phi_k, q \rangle = 0$ whenever $q \in \Pi_{k-1}$.

Proof. This follows since if $q \in \Pi_{k-1}$, then $q(x) = \sum_{i=0}^{k-1} \sigma_i \phi_i(x)$ for some $\sigma_i \in \mathbb{R}$, $i = 0, 1, \dots, k-1$, so

$$\langle \phi_k, q \rangle = \sum_{i=0}^{k-1} \sigma_i \langle \phi_k, \phi_i \rangle = 0. \quad \square$$

Remark: note from the above argument that if $q(x) = \sum_{i=0}^k \sigma_i \phi_i(x)$ is of exact degree k (so $\sigma_k \neq 0$), then $\langle \phi_k, q \rangle = \sigma_k \langle \phi_k, \phi_k \rangle \neq 0$.

Theorem. Suppose that $\{\phi_0, \phi_1, \dots, \phi_n, \dots\}$ is a set of orthogonal polynomials. Then, there exist sequences of real numbers $(\alpha_k)_{k=1}^\infty$, $(\beta_k)_{k=1}^\infty$, $(\gamma_k)_{k=1}^\infty$ such that a three-term recurrence relation holds of the form

$$\phi_{k+1}(x) = \alpha_k(x - \beta_k)\phi_k(x) - \gamma_k\phi_{k-1}(x), \quad k = 1, 2, \dots$$

Proof. The polynomial $x\phi_k \in \Pi_{k+1}$, so there exist real numbers

$$\sigma_{k,0}, \sigma_{k,1}, \dots, \sigma_{k,k+1}$$

such that

$$x\phi_k(x) = \sum_{i=0}^{k+1} \sigma_{k,i} \phi_i(x)$$

as $\{\phi_0, \phi_1, \dots, \phi_{k+1}\}$ is a basis for Π_{k+1} . Now take the inner product on both sides with ϕ_j where $j \leq k-2$. On the left-hand side, note $x\phi_j \in \Pi_{k-1}$ and thus

$$\langle x\phi_k, \phi_j \rangle = \int_a^b w(x)x\phi_k(x)\phi_j(x) dx = \int_a^b w(x)\phi_k(x)x\phi_j(x) dx = \langle \phi_k, x\phi_j \rangle = 0,$$

by the above lemma for $j \leq k-2$. On the right-hand side

$$\left\langle \sum_{i=0}^{k+1} \sigma_{k,i} \phi_i, \phi_j \right\rangle = \sum_{i=0}^{k+1} \sigma_{k,i} \langle \phi_i, \phi_j \rangle = \sigma_{k,j} \langle \phi_j, \phi_j \rangle$$

by the linearity of $\langle \cdot, \cdot \rangle$ and orthogonality of ϕ_i and ϕ_j for $i \neq j$. Hence $\sigma_{k,j} = 0$ for $j \leq k-2$, and so

$$x\phi_k(x) = \sigma_{k,k+1}\phi_{k+1}(x) + \sigma_{k,k}\phi_k(x) + \sigma_{k,k-1}\phi_{k-1}(x).$$

Almost there: taking the inner product with ϕ_{k+1} reveals that

$$\langle x\phi_k, \phi_{k+1} \rangle = \sigma_{k,k+1} \langle \phi_{k+1}, \phi_{k+1} \rangle,$$

so $\sigma_{k,k+1} \neq 0$ by the above remark as $x\phi_k$ is of exact degree $k+1$ (e.g., from above Gram-Schmidt notes). Thus,

$$\phi_{k+1}(x) = \frac{1}{\sigma_{k,k+1}}(x - \sigma_{k,k})\phi_k(x) - \frac{\sigma_{k,k-1}}{\sigma_{k,k+1}}\phi_{k-1}(x),$$

which is of the given form, with

$$\alpha_k = \frac{1}{\sigma_{k,k+1}}, \quad \beta_k = \sigma_{k,k}, \quad \gamma_k = \frac{\sigma_{k,k-1}}{\sigma_{k,k+1}}, \quad k = 1, 2, \dots$$

That completes the proof. □

Example. The inner product $\langle f, g \rangle = \int_{-\infty}^{\infty} e^{-x^2} f(x)g(x) dx$

gives orthogonal polynomials called the **Hermite polynomials**,

$$\phi_0(x) \equiv 1, \quad \phi_1(x) = 2x, \quad \phi_{k+1}(x) = 2x\phi_k(x) - 2k\phi_{k-1}(x) \text{ for } k \geq 1.$$

Listing 1: hermite_polys.m

```

1 %% Demonstrate Hermite Orthogonal Polynomials
2 lw = 'linewidth';
3 x = linspace(-2.2, 2.2, 256);
4 H_old = ones(size(x));
5 H = figure(1); clf;
6 plot(x, H_old, 'r-', lw,2)
7 set(get(H,'children'),'fontsize', 16);
8 hold on; pause
9
10 H = 2*x;
11 plot(x, H, lw,2, 'color',[0 0.75 0])
12 pause
13
14 for n = 1:4
15     % use the three-term recurrence
16     H_new = (2*x).*H - (2*n)*H_old;
17     plot(x, H_new, lw,2, 'color',rand(3,1))
18     pause;
19     H_old = H; H = H_new;
20 end
21 legend('H_0(x)', 'H_1(x)', 'H_2(x)', 'H_3(x)', 'H_4(x)', 'H_5(x)')
22 xlabel('x'); title('Hermite orthogonal polynomials')

```

