Part A: Differential Equations 2 HT2021

Problem Sheet 1

1. Reduction of order and Variation of Parameters.
Define
$$\mathfrak{L}y(x) \equiv x^2 y''(x) - x(x+2)y'(x) + (x+2)y(x), \quad 1 < x < 2.$$
Check that $y(x) = x$ is a solution of $\mathfrak{L}y = 0$ and use reduction of order to find the general solution. Hence solve the following problem by variation of parameters:
$$\mathfrak{L}y(x) = x^3, \qquad y(1) = 0, \quad y(2) = 0.$$

2. Green's function via Variation of Parameters.
Use variation of parameters to solve the problem
$$y''(x) - 2y'(x) + 2y(x) = f(x), \qquad y(0) = 0, \quad y\left(\frac{\pi}{2}\right) = 0, \qquad (\star)$$
where f is a given continuous function. Show that the solution can be written in the form
$$y(x) = \int_0^{\pi/2} g(x,\xi) f(\xi)\, \mathrm{d}\xi$$
and determine the Green's function g. Evaluate the integral when $f(x) = \mathrm{e}^x$ and check that the resulting expression for y does indeed satisfy $(\star)$.

3. Adjoint.
For each of the problems below, use the adjoint relation, $\langle \mathfrak{L}y, w\rangle \equiv \langle y, \mathfrak{L}^* w\rangle$, to determine the differential operator and boundary conditions for the adjoint problem. In each case state whether the operator and/or the full system is self-adjoint.
   (a) $\mathfrak{L}y = y''$, $\quad 2y(0) + y'(0) = 0, \quad y(1) + y'(1) = 0.$
   (b) $\mathfrak{L}y = y''$, $\quad 2y(0) + y'(1) = 0, \quad y(1) + y'(0) = 0.$
   (c) $\mathfrak{L}y = y'''' - y'$, $\quad y'(0) - y''(0) = 0, \quad y'''(0) = 0, \quad y(1) = 0, \quad y'(1) - y'''(1) = 0.$

4. FAT and Existence. Determine the parameter values (A, B) that yield existence of a solution for each of the following inhomogeneous BVPs.
   (a) For $0 \le x \le 2\pi$:
$$y''(x) + y(x) = A\sin x + B\cos x + 2\sin\left(x + \frac{\pi}{3}\right) + \sin^3 x, \qquad y(0) = y(2\pi), \qquad y'(0) = y'(2\pi).$$
   (b) For $0 \le x \le 1$:
$$y''(x) + 2y'(x) + y(x) = 1, \qquad y'(0) + y(0) = A, \qquad y'(1) + y(1) = 3.$$

[*Hint: Note that the problem in* (a) *is fully self-adoint. In* (b), *show that the homogeneous adjoint problem has solution* $w(x) = \mathrm{e}^x$.]

5. Computing Green's function. Obtain the Green's function for the following operators, using the delta function construction:

   (a) $\mathfrak{L}y = -y''$, $\quad 0 < x < 1$, $\quad y(0) - y'(1) = 0$, $\quad y(0) + y(1) = 0$.

   (b) $\mathfrak{L}y = y'' - y$, $\quad 0 < x < 2\pi$, $\quad y(0) - y(2\pi) = 0$, $\quad y'(0) - y'(2\pi) = 0$.

   In (b), what goes wrong if we change the operator to $\mathfrak{L}y = y'' + y$ (for the same boundary conditions)? Why?

6. Green's function for Initial Value Problem.
   Consider the inhomogeneous ODE

$$\mathfrak{L}y(x) = P_2(x)y''(x) + P_1(x)y'(x) + P_0(x)y(x) = f(x) \tag{$\dagger$}$$

   for $x > 0$, subject to initial conditions $y(0) = 0 = y'(0)$. Suppose that the homogeneous ODE $\mathfrak{L}y = 0$ has linearly independent solutions y_1 and y_2 satisfying $y_1(0) = 0 = y_2'(0)$.

   (a) Use variation of parameters to construct the solution y to ($\dagger$) and determine the Green's function g such that

$$y(x) = \int_0^\infty g(x, \xi) f(\xi)\, \mathrm{d}\xi.$$

   (b) State the ODE and boundary conditions satisfied by g, in terms of the delta-function, and show that this approach reproduces the expression for g from part (a).