Problem Sheet 2

1. Eigenfunction expansion.

   (a) Find the general solution of the Cauchy–Euler equation

   $$x^2 y''(x) + 3xy'(x) + (1+\alpha)y(x) = 0,$$

   where α is a given positive constant.

   (b) Use (a) to determine the eigenvalues λ_j and eigenfunctions y_j of the self-adjoint problem

   $$-\left(x^3 y'(x)\right)' = \lambda x y, \qquad y(1) = 0, \qquad y(\mathrm{e}) = 0.$$

   (c) Obtain the eigenfunction expansion for the solution of the inhomogeneous problem

   $$\left(x^3 y'(x)\right)' = x, \qquad y(1) = 0, \qquad y(\mathrm{e}) = 0.$$

   Give the coefficients explicitly, i.e. compute the integrals.

2. Sturm–Liouville form.
   Consider the general second order eigenvalue problem

   $$\mathfrak{L}y(x) = A(x)y''(x) + B(x)y'(x) + C(x)y(x) = \lambda y(x), \qquad a < x < b \qquad (\star)$$

   where $A(x), B(x), C(x)$ are given functions with $A(x) \neq 0$ for $x \in [a,b]$. Show that $(\star)$ can always be put into Sturm–Liouville form,

   $$-\left(p(x)y'(x)\right)' + q(x)y = \lambda r(x) y,$$

   and determine $p(x)$, $q(x)$, $r(x)$ in terms of $A(x)$, $B(x)$, $C(x)$.

   What orthogonality condition will the eigenfunctions satisfy?

3. Eigenvalue expansion — two routes.
   Consider the following eigenvalue problem on $0 \leq x \leq 1$:

   $$\mathfrak{L}y = y'' + 2y' + y = \lambda y, \qquad y'(0) + y(0) = 0, \qquad y'(1) + y(1) = 0.$$

   (a) Compute the eigenvalues λ_k, eigenfunctions y_k and adjoint eigenfunctions w_k.

   (b) Under what condition on f does a solution $y(x)$ exist for the inhomogeneous problem

   $$\mathfrak{L}y(x) = f(x) \quad (0 < x < 1), \qquad y'(0) + y(0) = 0, \qquad y'(1) + y(1) = 0?$$

   Assuming that this condition is satisfied:

   (i) obtain the coefficients in an eigenfunction expansion $y(x) = \sum_k^\infty c_k y_k(x)$;

   (ii) show that the eigenfunction expansion for the equivalent Sturm–Liouville problem matches the one you get in part (i).

4. Green's function for Sturm-Liouville. Consider the Sturm-Liouville operator

$$\mathfrak{L}y = -(py')' + qy, \quad a < x < b,$$

where $p(x) \neq 0$ on $a < x < b$, plus the boundary conditions

$$\mathfrak{B}_\ell y \equiv y(a) = 0, \qquad\qquad \mathfrak{B}_r y \equiv y(b) = 0.$$

Variation of parameters gives the following formula for the Green's function:

$$g(x,\xi) = \begin{cases} \dfrac{-y_\ell(x)y_r(\xi)}{W(\xi)p(\xi)} & a < x < \xi < b, \\ \dfrac{-y_\ell(\xi)y_r(x)}{W(\xi)p(\xi)} & a < \xi < x < b, \end{cases} \tag{†}$$

where $\mathfrak{L}y_\ell = 0 = \mathfrak{L}y_r$, $\mathfrak{B}_\ell y_\ell = 0 = \mathfrak{B}_r y_r$, and $W = y_\ell y_r' - y_\ell' y_r$ is the Wronskian.

(a) Re-derive equation (†) by constructing the Green's function satisfying $\mathfrak{L}_x g(x,\xi) = \delta(x-\xi)$.

(b) Obtain an alternative expression for the Green's function in terms of an eigenfunction expansion $g(x,\xi) = \sum_k c_k(\xi) y_k(x)$, where the y_k are eigenfunctions satisfying $\mathfrak{L}y_k = \lambda_k y_k$.

(c) Show that the two formulas agree by expanding (†) in an eigenfunction expansion and showing that the coefficients match, i.e. write $g(x,\xi) = \sum_k d_k(\xi) y_k(x)$ and show that $d_k \equiv c_k$.

5. Legendre's equation and the Fredholm Alternative. Consider *bounded* solutions of the eigenvalue problem

$$\mathfrak{L}y(x) = \left(1 - x^2\right) y''(x) - 2xy'(x) = \lambda y(x), \qquad -1 < x < 1. \tag{\#}$$

(a) Use the inner product relation to compute $\mathfrak{L}^*$ and show that the boundary terms vanish identically. Why are no boundary conditions given for (#)?

(b) Convert (#) to Sturm–Liouville form. What orthogonality relation do the eigenfunctions satisfy?

(c) Verify that $y_0(x) = 1$ is an eigenfunction for $\lambda_0 = 0$. For the inhomogeneous problem $\mathfrak{L}y(x) = f(x)$ to be solvable for y, what condition must f satisfy?

(d) Consider the equation $\mathfrak{L}y(x) = -2x$. Explain via the Fredholm Alternative why this problem should have a non-unique solution. Show that

$$y = x + A\log\left(\frac{1+x}{1-x}\right) + B$$

is a solution for any values of A and B. What can you conclude about the constant A?

(e) Find the general solution of $\mathfrak{L}y = 1$. Does this match your reasoning in (c)?