Part A: Differential Equations 2 **HT2021**

Problem Sheet 3

1. <u>Frobenius method.</u> Consider the differential equation

$$(x-1)y''(x) - xy'(x) + y(x) = 0$$

   (a) Determine the appropriate form of the series expansion about $x = 1$ for two linearly independent solutions. [*You do not need to compute the coefficients.*]

   (b) Use (a) to obtain one closed form solution (i.e. not in the form of an infinite series). [*Hint: Consider the choice of coefficients in the second Frobenius series.*]

2. <u>The point $x = \infty$.</u> Consider the differential equation

$$x^3 y''(x) + y(x) = 0. \qquad (\star)$$

   (a) Use the transformation of variables $x = 1/t$ to show that (**??**) has a regular singular point at $x = \infty$ and determine the indicial exponents.

   (b) Obtain the first Frobenius solution in the form of an infinite series in powers of t, i.e. solve explicitly for the coefficients.

   (c) Find the form of the second Frobenius solution and obtain (but do not attempt to solve) a recurrence relation for the coefficients in the series.

3. <u>Bessel functions.</u> Consider *Bessel's differential equation* (of order n):

$$x^2 y''(x) + xy'(x) + (x^2 - n^2)y(x) = 0, \qquad (\dagger)$$

   for integer $n > 0$.

   (a) Find the indicial exponents α_1, α_2 (with $\operatorname{Re}\alpha_1 > \operatorname{Re}\alpha_2$) for the local series expansion of (**??**) about $x = 0$.

   (b) Determine the series $y(x) = \sum_{k=0}^{\infty} a_k x^{k+\alpha_1}$ that solves (**??**), giving the coefficients a_k in closed form. Find a_0 such that the series is the expansion of the *Bessel functions of first kind*,

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{\left(-x^2/4\right)^k}{k!(k+n)!}. \qquad (\#)$$

   (c) Using (**??**), show that the following recursion relation is true for all integers $n \geq 0$:

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x).$$

   (d) For any integer $n \geq 0$, show that

$$\int_0^1 x\,[J_n(\alpha x)]^2\,\mathrm{d}x = \frac{1}{2}\left[J_n'(\alpha)\right]^2,$$

   where α is a zero of J_n. [*Hint: Substitute $z = \alpha x$, integrate by parts, and use the fact that J_n satisfies Bessel's equation.*]

4. Bessel functions in a Sturm-Liouville problem.

   (a) Determine the *bounded* eigenfunctions y_j and eigenvalues λ_j of the following singular Sturm–Liouville problem on $0 \le x \le 1$:

$$-\big(xy'(x)\big)' = \lambda x y(x), \qquad y(1) = 0.$$

   Hint: Use a change of variables of the form $r = \beta x$.

   (b) Use (a) to obtain the eigenfunction expansion for the *bounded* solution of the following inhomogeneous problem on $0 \le x \le 1$:

$$\big(xy'(x)\big)' = x, \qquad y(1) = 0.$$

   Leave the coefficients c_k in your final answer in terms of integrals containing Bessel functions.

5. Legendre functions and associated Legendre functions. Consider Legendre's equation

$$(1-x^2)y''(x) - 2xy'(x) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right) y(x) = 0,$$

and let $P_\ell^m(x)$ denote the solution for integers $0 \le m \le \ell$. Show that

$$\int_{-1}^{1} P_k^m(x) P_\ell^m(x)\,\mathrm{d}x = \begin{cases} 0 & \text{if } \ell \ne k \\ \dfrac{2}{(2k+1)}\dfrac{(k+m)!}{(k-m)!} & \text{if } \ell = k. \end{cases}$$

[*You may use without proof Rodrigues' formula given in lectures, and also the identity*

$$\int_{-1}^{1} \left(1-x^2\right)^\ell \,\mathrm{d}x = \frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!},$$

or for extra fun try to show this as well...]