Part A: Differential Equations 2 HT2021

Problem Sheet 4

1. Order relations.

   (a) Determine the order as $\epsilon \to 0$ of
   $$\frac{\sqrt{\epsilon}}{1-\cos\epsilon}.$$

   (b) Obtain an asymptotic expansion of $\exp(\tan\epsilon)$ in integer powers of ϵ up to order $O(\epsilon^4)$.

   (c) Show that $\log\epsilon = o(\epsilon^{-p})$ as $\epsilon \to 0^+$ for all $p > 0$.

2. Roots. Find expansions for all roots of the equations below as $\epsilon \to 0$ with two nonzero terms in each expansion:

   (a) $\epsilon x^3 - x + 1 = 0$,

   (b) $\tan x = \dfrac{\epsilon}{x}$.

   [*Hint: Roots are near $n\pi$ for integers n. The root near zero must be treated separately, and requires the balance $x = O\left(\epsilon^{1/2}\right)$ (can you see why?).*]

3. Regular perturbation. Find the first two terms in an asymptotic expansion in powers of the small parameter ϵ of the solution of
   $$xy'(x) + y(x) = \epsilon y(x)^{1/2}, \qquad x > 1, \quad y(1) = 1.$$
   Explain why the expansion is not valid as $x \to \infty$. What form of rescaling would be necessary to examine the behaviour for large x?
   [*You do not need to carry out the resulting analysis.*]

4. Inner and outer expansions. Find inner and outer expansions, correct up to and including terms of $O(\epsilon)$, for the function
   $$f(x;\epsilon) = \frac{\mathrm{e}^{-x/\epsilon}}{x} + \frac{\sin x}{x} - \coth x,$$
   for $x > 0$ and $0 < \epsilon \ll 1$. Compare the inner and outer approximations with the exact function f by plotting all three on the same graph for various small values of ϵ.

5. Singular perturbation. Use matched asymptotic expansions to find leading-order outer and inner solutions to the boundary value problem
   $$\epsilon y''(x) + xy'(x) + y(x) = 0, \qquad 1 < x < 2,$$
   $$y(1) = 0, \qquad y(2) = 1,$$
   where $0 < \epsilon \ll 1$. Form the leading-order composite solution.
   Explain briefly why there is a boundary layer at $x = 1$ and not at $x = 2$.
   How might the analysis change if ϵ were small and *negative*? [*You do not need to find the solution in this case.*]

6. Singular perturbation 2. Construct leading-order inner and outer solutions to

$$\epsilon u''(x) + u'(x) = \frac{u(x) + u(x)^3}{1 + 3u(x)^2}, \qquad 0 < x < 1,$$
$$u(0) = 0, \qquad u(1) = 1$$

where $0 < \epsilon \ll 1$. [*You will only be able to determine the outer solution implicitly.*]