

Part A Integration

Problem Sheet 3: Convergence theorems and consequences

Relevant video lectures: 4C 4D 5A 5B 5C 6A 6B 6C

Note: *Answers to this problem sheet should, if possible, be carefully justified by reference to theorems in the lectures and by showing that the conditions of those theorems are satisfied.*

1. Let $\alpha \in (0, \infty)$. Show that $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x} dx$ exists as a Lebesgue integral, and that $\Gamma(\alpha + 1) = \alpha \Gamma(\alpha)$.
2. We know that $\frac{\sin x}{x}$ is not integrable over $(1, \infty)$. Deduce, or prove otherwise, that neither of the following functions is integrable over the given intervals:

$$(i) \quad \frac{\sin(x^2)}{x} \quad \text{over} \quad (1, \infty), \quad (ii) \quad \frac{1}{x} \sin \frac{1}{x^2} \quad \text{over} \quad (0, 1).$$

3. By comparing terms in binomial expansions, or otherwise, show that $(1 + \frac{x}{n})^n \leq (1 + \frac{x}{n+1})^{n+1}$ for $n \geq 2$, $x \geq 0$. Use the MCT to deduce that

$$\lim_{n \rightarrow \infty} \int_1^2 \left(1 + \frac{x}{n}\right)^{-n} dx = e^{-1} - e^{-2}.$$

- *4. Use the binomial expansion of $(1 - x)^{-k}$ to show that $(1 - \frac{u}{n})^{-n} \geq (1 - \frac{u}{n+1})^{-(n+1)}$ for $n \geq 2$ and $0 \leq u \leq n$. Hence, or otherwise, prove that

$$\lim_{n \rightarrow \infty} n^\alpha \int_0^1 x^{\alpha-1} e^{-n\beta x} (1 - x)^n dx = (\beta + 1)^{-\alpha} \Gamma(\alpha)$$

where $\alpha > 0$, $\beta > -1$, and $\Gamma(\alpha) = \int_0^\infty e^{-u} u^{\alpha-1} du$.

5. Show that $0 \leq \frac{x}{1+x^\alpha} \leq 1$ for $\alpha > 1$, $x \geq 0$, and deduce that

$$\lim_{n \rightarrow \infty} \int_0^{2\pi} \frac{nx \sin x}{1 + n^\alpha x^\alpha} dx = 0.$$

- *6. Prove that $\lim_{n \rightarrow \infty} \int_0^{n^2} n \left(\sin \frac{x}{n} \right) e^{-x^2} dx = \frac{1}{2}$.

7. Let $\alpha > -1$. Show that $x^\alpha \log x$ is integrable over $(0, 1)$, and

$$\int_0^1 x^\alpha \log x \, dx = -(1 + \alpha)^{-2}.$$

Deduce that for $\beta > -1$, $x^\beta(1-x)^{-1} \log x$ is integrable over $(0, 1)$, and

$$\int_0^1 x^\beta(1-x)^{-1} \log x \, dx = -\sum_{n=1}^{\infty} (n + \beta)^{-2}.$$

8. Show that for $n > 0$, $e^{-nx} \sin x$ is integrable over $[0, \infty)$, and

$$\int_0^{\infty} e^{-nx} \sin x \, dx = \frac{1}{1 + n^2}.$$

Deduce that for $0 \leq a \leq 1$, $(e^x - a)^{-1} \sin x$ is integrable over $[0, \infty)$, and

$$\int_0^{\infty} (e^x - a)^{-1} \sin x \, dx = \sum_{n=1}^{\infty} \frac{a^{n-1}}{1 + n^2}.$$

- *9. Let $a \in (0, 1)$. Prove that $\frac{e^{-ax} - e^{(a-1)x}}{1 - e^{-x}}$ is an integrable function on \mathbb{R} , and that

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{e^{-ax} - e^{(a-1)x}}{1 - e^{-x}} \, dx = \frac{1}{a} + \frac{1}{a-1} + \frac{1}{a+1} + \frac{1}{a-2} + \frac{1}{a+2} + \frac{1}{a-3} + \dots$$

10. Prove that $\int_0^{\infty} \cos x \arctan(\lambda x) e^{-x} \, dx \rightarrow \frac{\pi}{4}$ as $\lambda \rightarrow \infty$.

11. Let $J_0(x) = \frac{2}{\pi} \int_0^{\pi/2} \cos(x \cos \theta) \, d\theta$. Show that J_0 is differentiable on \mathbb{R} .

*Show that the function Γ , as defined in Q.1, is differentiable on $(0, \infty)$.

12. Let $f(x, y) = y^3 e^{-y^2 x}$, $F(y) = \int_0^{\infty} f(x, y) \, dx$. Calculate $F'(0)$ and $\int_0^{\infty} \frac{\partial f}{\partial y}(x, 0) \, dx$.
How do your answers relate to the theorem about differentiating through integrals?

13. By differentiating through the integral sign, evaluate the following integrals:

(i) $\int_0^{\infty} \frac{e^{-x} \sin tx}{x} \, dx,$

*(ii) $\int_0^{\pi/2} \log(a^2 \cos^2 x + b^2 \sin^2 x) \, dx$, where $a, b > 0$.

- *14. Let $K(t) = \int_1^{\infty} \frac{\cos(tx)}{x^2} \, dx$. Show carefully that $K'(t) = \frac{K(t) - \cos t}{t}$ for $t > 0$.