Problem Sheet 4: Fubini's Theorem, L^p-spaces

1. Evaluate $\int_0^1 \left(\int_0^x e^{-y}\,dy\right)\,dx$ and $\int_0^1 \left(\int_0^{x-x^2} (x+y)\,dy\right)\,dx$
   (a) directly;
   (b) by reversing the order of integration.

2. In each of the following cases, is f integrable over the given region? [Give careful justification.]
   (i) $f(x,y) = e^{-xy}$ over $[0,\infty)\times[0,\infty)$;
   (ii) $f(x,y) = e^{-xy}$ over $\{(x,y) : 0 < x < y < x + x^2\}$;
   (iii) $f(x,y) = \dfrac{(\sin x)(\sin y)}{x^2+y^2}$ over $(-\frac{\pi}{2},\frac{\pi}{2})\times(-\frac{\pi}{2},\frac{\pi}{2})$.

3. [Applications of Tonelli or Fubini should be carefully justified.]
   (a) Let $J_0(x) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} \cos(x\cos\theta)\,d\theta$. Show that $\displaystyle\int_0^\infty J_0(x)e^{-ax}\,dx = \frac{1}{\sqrt{1+a^2}}$ if $a > 0$.
   (b) Take $b > a > 1$. By considering x^{-y} over $(1,\infty)\times(a,b)$, show that $\displaystyle\int_1^\infty \frac{x^{-a}-x^{-b}}{\log x}\,dx$ exists, and find its value.

4. Let $f \in \mathcal{L}^1(\mathbb{R})$ be non-negative with $\int_{-\infty}^\infty f(x)\,dx = 1$, and let $F(x) = \int_{-\infty}^x f(y)\,dy$. Assume that $xf(x) \in \mathcal{L}^1(\mathbb{R})$. Use Fubini's Theorem to prove that

$$\int_0^\infty (1 - F(x))\,dx = \int_0^\infty xf(x)\,dx, \qquad \int_{-\infty}^0 F(x)\,dx = -\int_{-\infty}^0 xf(x)\,dx.$$

   Now let g be a bounded measurable function, and let

$$G(y) = \int_{\{g(x)\le y\}} f(x)\,dx.$$

   Prove that

$$\int_0^\infty (1 - G(y) - G(-y))\,dy = \int_{-\infty}^\infty f(x)g(x)\,dx.$$

   [*Remark (not a hint): Imagine that f is the probability density function of a random variable X. The first part of the question then says that $\mathbb{E}(X) = \int_0^\infty (\mathbb{P}[X > x] - \mathbb{P}[X \le -x])\,dx$. This formula holds for all random variables (discrete, continuous, etc) with $\mathbb{E}(|X|) < \infty$. In particular it holds for $g(X)$. Then the last part proves that $\mathbb{E}[g(X)] = \int_{-\infty}^\infty f(x)g(x)\,dx$, a fact sometimes known as the Law of the Unconscious Statistician.*]

5. (a) Let $\alpha > 1$ and $f(x,y) = (x^2+y^2)^{-\alpha}$ and $g(x,y) = (1+x^2+y^2)^{-\alpha}$. Show that f is integrable over $[1,\infty)\times[0,\infty)$ [Hint: Change of variables $y = ux$ may help]. Deduce that f is integrable over $[0,1]\times[1,\infty)$, and that g is integrable over $\mathbb{R}^2$.

   (b) Use polar coordinates to show that g is integrable over $\mathbb{R}^2$.

6. For $p > 0$, calculate $\|f\|_p$ when f is (i) $\chi_{(0,1)}$, (ii) $\chi_{(1,2)}$, (iii) $\chi_{(0,2)}$.

   Now assume that $0 < p < 1$. Is $\|\cdot\|_p$ a norm on L^p?

   For $f, g \in L^p(\mathbb{R})$, let $d_p(f,g) = \int |f-g|^p$. Show that d_p is a metric on $L^p(\mathbb{R})$.

7. Consider the relation $\sim$ on the space of measurable functions $f : \mathbb{R} \to \mathbb{R}$ given by: $f \sim g \iff f = g$ a.e.

   State which properties of null sets are used to prove each of the following true statements (f, g, h, etc are measurable functions):

   (i) $f \sim f$,
   (ii) $f \sim g \implies g \sim f$,
   (iii) $f \sim g, g \sim h \implies f \sim h$,
   (iv) If $f_n \sim g_n$ for all $n \in \mathbb{N}$, then $\sup f_n \sim \sup g_n$,
   (v) If $f \sim g$, then $h \circ f \sim h \circ g$.

   *Give an example where h is injective, $f \sim g$, but $f \circ h \not\sim g \circ h$.

8. Let $p > 1$. Give examples of sequences (f_n) and (g_n) in $L^p(0,1)$ such that

   (i) $\lim_{n\to\infty} f_n(x) = 0$ a.e. but $\lim_{n\to\infty} \|f_n\|_p \neq 0$;
   (ii) $\lim_{n\to\infty} \|g_n\|_p = 0$ but $\lim_{n\to\infty} g_n(x)$ does not exist for any $x \in (0,1)$.

   For each $\varepsilon > 0$ find a measurable subset E_ε of $[0,1]$ such that $m(E_\varepsilon) < \varepsilon$ and $f_n(x) \to 0$ uniformly on $[0,1] \setminus E_\varepsilon$.

   Find a subsequence (g_{n_r}) such that $\lim_{r\to\infty} g_{n_r}(x) = 0$ a.e.

*9. A function $g : [0,\infty) \to \mathbb{R}$ is *convex* if

$$g(x) = \sup\{\alpha x + \beta : \alpha y + \beta \leq g(y) \text{ for all } y \in [0,\infty)\}.$$

   If g is continuous on $[0,\infty)$ with non-negative second derivative on $(0,\infty)$, then g is convex.

   Let $f : [0,1] \to [0,\infty)$ be bounded and measurable, $M_n \int_0^1 f^n \, dx = \|f\|_{L^n}^n$, and $\|f\|_{L^\infty} = \inf\{\gamma > 0 : f(x) \leq \gamma \text{ a.e.}\} < \infty$. Show that

   (i) $g\left(\int_0^1 f(x)\, dx\right) \leq \int_0^1 g(f(x))\, dx$ for every convex function g;
   (ii) $M_n^2 \leq M_{n+1} M_{n-1}$;
   (iii) $\|f\|_{L^n} \leq M_{n+1}/M_n \leq \|f\|_{L^\infty}$;
   (iv) $\lim_{n\to\infty} M_{n+1}/M_n = \|f\|_{L^\infty}$.