Part A Integration

Supplementary Problem Sheet:

Note: Q.1 and Q.5 go somewhat beyond the syllabus for A4. Q.2, Q.3 and Q.4 are extracted from past exam papers; Q.4 is almost the entire exam question.

1. (a) Let $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to [0, \infty)$ be Borel-measurable functions, and μ be a measure on $(\mathbb{R}, \mathcal{M}_{\text{Bor}})$. For $B \in \mathcal{M}_{\text{Bor}}$, let

$$(g_*\mu)(B) = \mu(g^{-1}(B)), \qquad (h \cdot \mu)(B) = \int_B h \, d\mu.$$

Show that $g_*\mu$ and $h \cdot \mu$ are measures on $(\mathbb{R}, \mathcal{M}_{\text{Bor}})$.

Let $f : \mathbb{R} \to [0, \infty]$ be Borel-measurable. Show that

$$\int_{\mathbb{R}} (f \circ g) \, d\mu = \int_{\mathbb{R}} f \, d(g_*\mu), \qquad \int_{\mathbb{R}} fh \, d\mu = \int_{\mathbb{R}} f \, d(h \cdot \mu).$$

[*Consider first $f = \chi_B$, then consider simple functions, and then apply the MCT.*]

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be an increasing bijection with a continuous derivative. Show that the measure $g_*(g'.m)$ is Lebesgue measure m on $\mathcal{M}_{\text{Bor}}$. [*You may assume that m is the unique measure μ on $(\mathbb{R}, \mathcal{M}_{\text{Bor}})$ such that $\mu(I) = b - a$ whenever I is an interval with endpoints a, b.*]

Let $f : \mathbb{R} \to [-\infty, \infty]$ be Borel-measurable. Show that f is integrable (with respect to m) if and only if $(f \circ g)g'$ is integrable, and then their integrals are equal.

2. A function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(t) = \int_0^\infty e^{-x^2} \cos\left(2xt^2\right) \, dx.$$

You may assume that this integral exists for all $t \in \mathbb{R}$ and that $f(0) = \sqrt{\pi}/2$.

Find an explicit formula for $f(t)$ for all $t \in \mathbb{R}$. You should justify your arguments carefully, making clear statements of any standard results that you use.

3. For $x > 0$ and $y > 0$, let
$$f(x,y) = \frac{1}{(1+y)(1+x^4y)}.$$
Show carefully that f is integrable over $(0,\infty) \times (0,\infty)$.

Hence or otherwise show that the following integral exists, and find its value:
$$\int_0^\infty \frac{\log x}{x^4-1}\,dx.$$
[*The formula*
$$f(x,y) = \frac{1}{x^4-1}\left(\frac{x^4}{1+x^4y} - \frac{1}{1+y}\right) \qquad (x \neq 1)$$
may be useful.]

4. (a) Let f be an integrable function on $(0,\infty)$ with $f(x) > 0$ for all $x > 0$, and let $a > 1$. For $x > 0$ and $y > 0$, let
$$g(x,y) = f(xy) - af(axy).$$
Show that
$$\int_0^1 g(x,y)\,dx = -\frac{1}{y}\int_y^{ay} f(t)\,dt < 0 \quad \text{whenever } y > 0,$$
and
$$\int_1^\infty g(x,y)\,dy > 0 \text{ whenever } x > 0.$$
Deduce that g is not integrable over $(0,1) \times (1,\infty)$.

(b) *In this part of the question, you may assume that* $e^{-x^2}x^3$ *is integrable over* $(0,\infty)$ *with integral* $1/2$.

For $x > 0$ and $y > 0$, let
$$h(x,y) = e^{-(x^2+y^2)}x^{3/2}y^{1/2}.$$
Show that h is integrable over $(0,\infty) \times (0,\infty)$.

Hence, or otherwise, show that
$$\int_0^{\pi/2} (\cos\theta)^{3/2}(\sin\theta)^{1/2}\,d\theta = \frac{1}{2}\left(\int_0^\infty e^{-u}u^{1/4}\,du\right)\left(\int_0^\infty e^{-v}v^{-1/4}\,dv\right).$$

5. Let E_n be measurable subsets of $\mathbb{R}$ with $m(E_n) \leq 2^{-n}$ for $n = 1, 2, \ldots$. Show that $\lim_{n\to\infty} \chi_{E_n}(x) = 0$ a.e.

Let $f \in \mathcal{L}^1(\mathbb{R})$. Show that $\lim_{n\to\infty} \int_{E_n} |f| = 0$. Deduce that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\int_E |f| < \varepsilon$ for all measurable sets E with $m(E) < \delta$.

Let $F(x) = \int_{-\infty}^x f(y)\,dy$. Show that F is absolutely continuous.