A3: Rings and Modules

Sheet I — HT21

I.1. Suppose that R with the binary operation $+$ is a (possibly non-commutative) group; $\times$ is an associative binary operation on R with an identity; and that $\times$ distributes over $+$. Show that R equipped with $+$ and $\times$ is a ring.

I.2. Suppose that V is a commutative group with all elements of finite order but some elements of arbitrarily larger order, meaning for all $n \in \mathbb{N}$ there is an element $x \in V$ of order at least n. Show that V is *not* the additive group of a ring.

I.3. Write $C(\mathbb{R})$ for the vector space of continuous functions $\mathbb{R} \to \mathbb{R}$ with pointwise addition. Is $C(\mathbb{R})$ a ring when equipped with functional composition as multiplication?

Show that $C(\mathbb{R})$ is a ring when equipped with pointwise addition and pointwise multiplication, and that $f \in C(\mathbb{R})$ is a zero divisor if and only if there is an interval on which f is identically 0.

I.4. Show that the set T of rational numbers with odd denominator is a subring of $\mathbb{Q}$. What are the units of T? What are the ideals of T?

I.5. Show that there are no ring homomorphisms $\mathbb{C} \to \mathbb{R}$, $\mathbb{R} \to \mathbb{Q}$, or $\mathbb{Q} \to \mathbb{Z}$.

I.6. Show that for $A \in M_n(\mathbb{R})$ there is some $p \in \mathbb{R}[X]$ such that $\mathbb{R}[A]$ is isomorphic to $\mathbb{R}[X]/\langle p\rangle$. Show that if $p, q \in \mathbb{R}[X]$ are quadratics with two distinct real roots then $\mathbb{R}[X]/\langle p\rangle \cong \mathbb{R}[X]/\langle q\rangle$. Justify which, if any, of $\mathbb{R}[A]$, $\mathbb{R}[B]$, $\mathbb{R}[C]$, and $\mathbb{R}[D]$ are isomorphic as rings where

$$A := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, B := \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}, C := \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \text{ and } D := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

I.7. Show that $\mathbb{Z}[X] \to \mathbb{Z}^2; p \mapsto (p(0), p(1))$ is a surjective ring homomorphism *i.e.* $\mathbb{Z}^2$ is a quotient of $\mathbb{Z}[X]$. Show that there is no surjective ring homomorphism $\mathbb{Z}[X] \to \mathbb{Z}^n$ for $n \geq 3$ *i.e.* $\mathbb{Z}^n$ is *not* a quotient of $\mathbb{Z}[X]$ for any $n \geq 3$.

I.8. Write $\mathrm{Int}(\mathbb{Z})$ for the set of polynomials mapping the integers to the integers. Show that $\mathrm{Int}(\mathbb{Z})$ is a subring of $\mathbb{Q}[X]$, but that $\mathrm{Int}(\mathbb{Z})$ and $\mathbb{Z}[X]$ are not isomorphic as rings.

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I.9. Show that if R is an integral domain that is a finite dimensional vector space over a field such that $R \to R; x \mapsto ax$ is linear for all $a \in R$, then R is a field. Show that

$$\mathbb{H} := \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}$$

is a subring of $M_2(\mathbb{C})$. Show that $\mathbb{H}$ is a 2-dimensional vector space over $\mathbb{C}$ in a way that makes right multiplication linear. Does it have any non-zero zero-divisors?

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