A3: Rings and Modules

Sheet IV — HT21

IV.1. Show that a cyclic submodule of a commutative ring R is a principal ideal. Suppose that $R := \mathbb{Z}[X]$ and $M := \langle 2, X\rangle$ as an R-module. Show that M is *not* a direct sum of cyclic $\mathbb{Z}[X]$-modules.

IV.2. Suppose that R is a commutative ring. Show that if every submodule of R (considered as an R-module) is free then R is a PID.

IV.3. Show that the maps

$$e_k : \mathbb{N}_0 \to \mathbb{Z}; x \mapsto \frac{x(x-1)\cdots(x-k+1)}{k!}$$

for $k \in \mathbb{N}_0$ (with e_0 identically 1) form a basis for $\mathrm{Int}(\mathbb{Z})$ as a $\mathbb{Z}$-module.

IV.4. Both the conclusions below follow from the Structure Theorem in the lectures notes. The point of this question is to see how the proof simplifies in each case.

(a) Suppose that R is a PID and M is a finitely generated R-module such that $\mathrm{Ann}_R(x) = \{0\}$ for all $x \neq 0_M$. Show that M is free.

(b) Suppose that G is a finitely generated commutative group. Show that G is isomorphic to a direct sum of cyclic groups.

IV.5. Suppose that G is a finite commutative group. Show that there is an element $x \in G$ such that the order of every $y \in G$ divides the order of x. Hence or otherwise show that $U(\mathbb{F})$ is a cyclic group when $\mathbb{F}$ is a finite field.

IV.6. Assuming that $N(a+bi) := a^2 + b^2$ is a Euclidean function on the Gaussian integers $\mathbb{Z}[i]$, put the following matrix into Smith Normal Form:

$$A := \begin{pmatrix} 13 & 2+2i \\ 3+i & i \end{pmatrix}.$$

IV.7. Let G be the commutative group with generators a, b, and c and relations $2a-16b-8c = 0$, and $4a+24b+8c = 0$. Find $s, r \in \mathbb{N}_0$, natural numbers $d_r \mid \cdots \mid d_1$, and an isomorphism $G \to (\mathbb{Z}/d_r\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/d_1\mathbb{Z}) \oplus \mathbb{Z}^s$.

IV.8. Derive the rational canonical form of the below matrix A, by putting the matrix $xI - A$ into Smith Normal Form.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 3 & -1 & 4 \\ 1 & 1 & -1 & 3 \end{pmatrix}$$

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