2. Discrete-time models for a single species

(Ruth Baker notes Chapter 1).

We now consider the case of a single species where the generations are *discrete* (no overlapping). Keeping in place all the other assumptions from the continuous time case, our model takes the form

$$N_{t+1} = f(N_t) = N_t g(N_t), \tag{2.1}$$

where N_t is the population density (biomass) at generation t, and $t = 0, 1, 2....$

2.1 Examples

Exponential growth

A simple example is

$$N_{t+1} = rN_t. \tag{2.2}$$

This is a linear difference (discrete) equation and we follow the standard method and look for a solution $N_t = \alpha\lambda^t$, where α and λ are constants. Substituting this into Equation (2.2), we have:

$$\alpha\lambda^{t+1} = \alpha\lambda^t r. \tag{2.3}$$

Hence, $\lambda = r$. Furthermore, $\alpha = N_0$, the initial population density, which is given.

Therefore:

$$N_t = r^t N_0 \to \begin{cases} \infty & r > 1 \\ N_0 & r = 1 \\ 0 & r < 1 \end{cases} . \tag{2.4}$$

Here, N_0 is the initial condition.

This is a bit like the corresponding continuous time model - it is unrealistic.

Discrete Logistic Growth Model

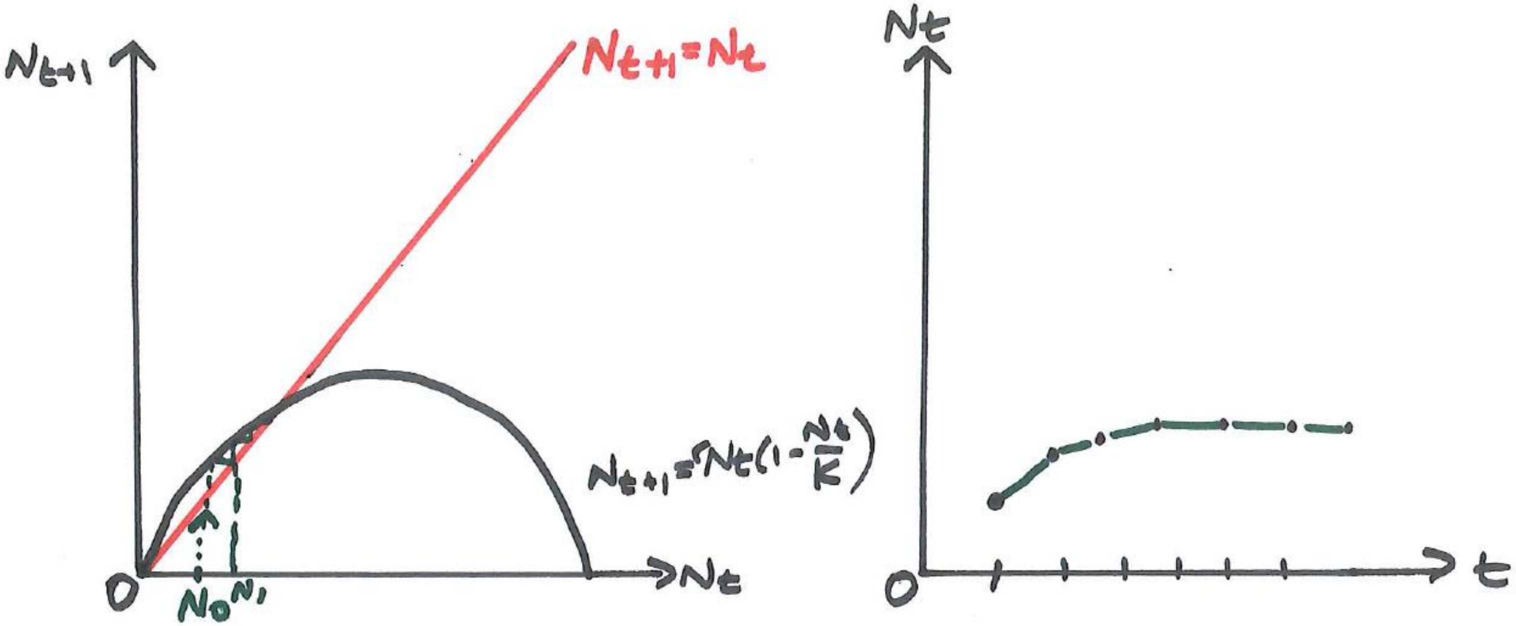
$$N_{t+1} = N_t \left[r \left(1 - \frac{N_t}{K} \right) \right], \quad r > 0 \quad K > 0. \tag{2.5}$$

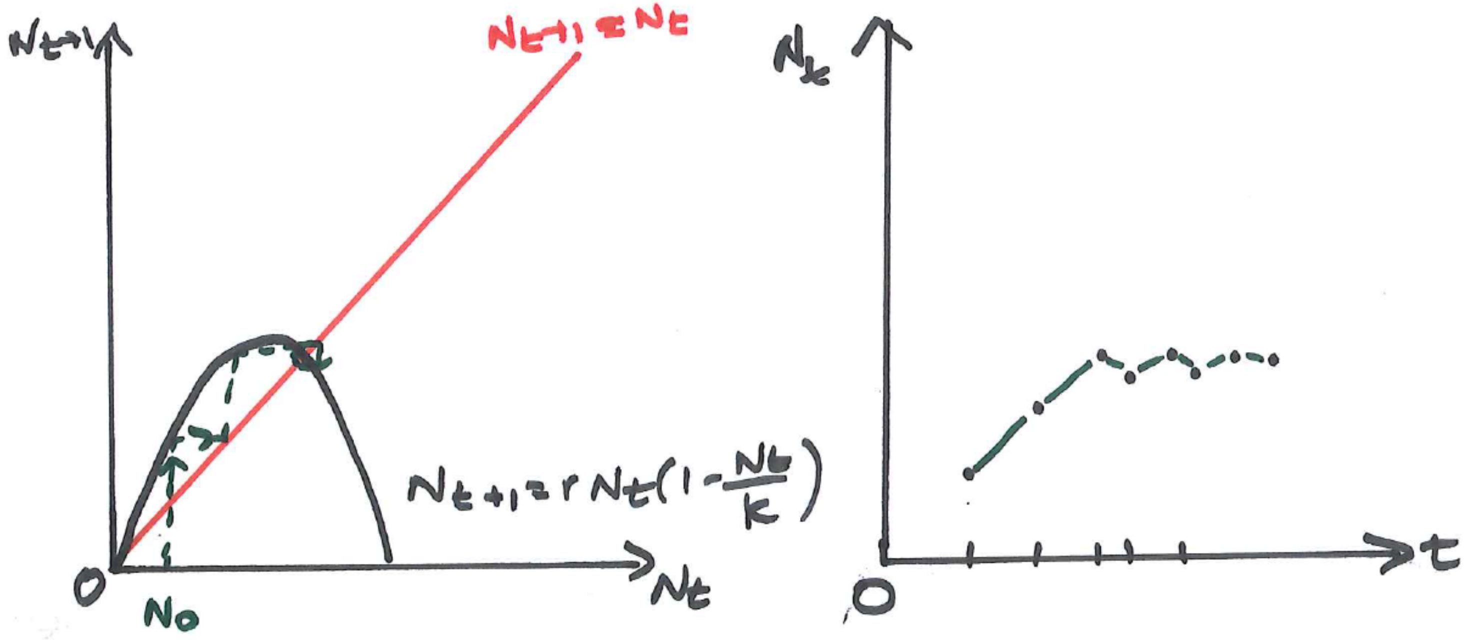
A *steady state*, N_s, satisfies

$$N_s = f(N_s) = N_s g(N_s). \tag{2.6}$$

So, here, $N_s = 0, N_s = (1 - \frac{1}{r})K$ are steady states. From a biological viewpoint, the second steady state only makes sense if $r \geq 1$.

2.2 Cobwebbing





Nt+1
Nt+1 = Nt
Nt+1 = r Nt(1 - Nt/K)
0
N0
Nt
Nt
0
t

2.3 Linear stability

To linearise about a steady state, we set

$$N_t = N_s + n_t, \tag{2.7}$$

where N_s is the steady state, and n_t is small. Note that N_s is time independent and satisfies $N_s = f(N_s)$. Hence

$$N_{t+1} = N_s + n_{t+1} = f(N_s + n_t) = f(N_s) + n_t f'(N_s) + O(n_t^2). \tag{2.8}$$

Using the definition of steady state, we have (ignoring higher order terms)

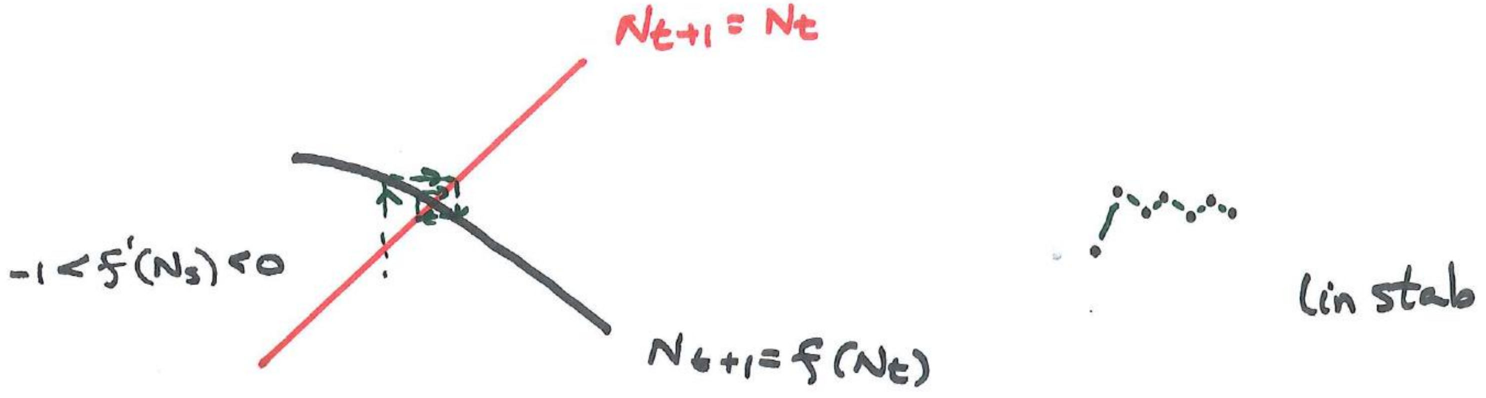
$$n_{t+1} = f'(N_s)n_t, \tag{2.9}$$

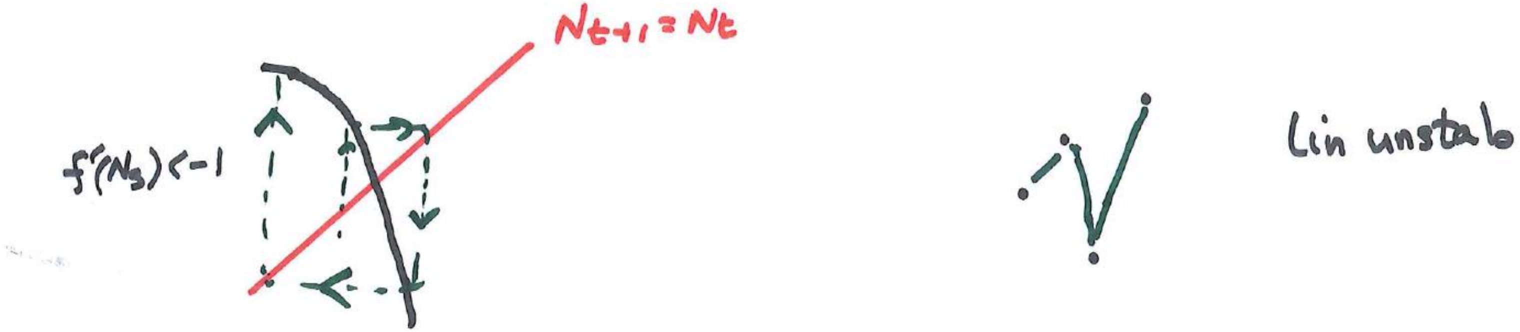
where $f'(N_s)$ is a constant, independent of t, and thus

$$n_t = \left[f'(N_s)\right]^t n_0. \tag{2.10}$$

This means that N_s is linearly stable if $|f'(N_s)| < 1$ and linearly unstable if $|f'(N_s)| > 1$.

Let us view this through the context of cobwebbing.


$N_{t+1} = N_t$
$-1 < f'(N_s) < 0$
$N_{t+1} = f(N_t)$
lin stab


$N_{t+1} = N_t$
$f'(N_s) < -1$
lin unstab


$N_{t+1} = N_t$
$f'(N_s) = -1$
neutrally stable

A bifurcation occurs at $|f'(N_s)| = 1$. This leads to two possibilities: $f'(N_s) = 1$ (*tangent bifurcation*); $f'(N_s) = -1$ (*pitchfork bifurcation*).

For the pitchfork bifurcation: $n_{t+1} = -n_t$, hence we have a *period 2* oscillation.

Summary

- Linear stability analysis
- Cobwebbing

End of Lecture 2_1

Summary of previous part

- Linear stability analysis
- Cobwebbing

2.3.1 Discrete time Logistic Model

Non-dimensionalising:

$$u_{t+1} = ru_t\,(1 - u_t) = f(u_t), \tag{2.11}$$

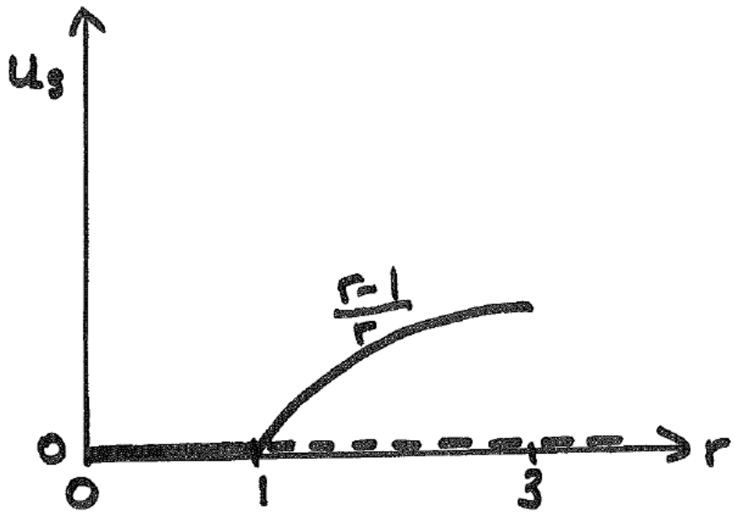
where $N_t = Ku_t$. Steady states: $u_s = 0, \frac{r-1}{r}$. Linear stability: $f'(u_s) = r - 2ru_s$.

Therefore $f'(0) = r$, $f'(\frac{r-1}{r}) = 2 - r$.

So, for $0 < r < 1$, $u_s = 0$ is linearly stable, $u_s = \frac{r-1}{r}$ is linearly unstable (and not biologically realistic).

While, for $1 < r < 3$, $u_s = 0$ is linearly unstable, $u_s = \frac{r-1}{r}$ is linearly stable (and biologically realistic).

A bifurcation occurs at $r = 1$.



What happens for $r \geq 3$?

At $r = 3$, $f'(\frac{r-1}{r}) = 2 - r = -1$ so we expect an oscillatory solution. As r gets bigger than 3, this non-zero steady state goes unstable (so, we have a bifurcation at $r = 3$).

Definition: The *trajectory*, or *orbit*, generated by u_0 is the set of points $u_0, u_1, u_2, ...$

We say that a point is *periodic of period* m (or m*-periodic*) if $f^m(u_0) = u_0, f^i(u_0) \neq u_0, i = 1, 2, 3..., m-1$, where $f^m(u)$ means "perform the operation f m times".

Therefore, for logistic growth, to investigate the period 2 solution, we set: $u_{t+2} = f^2(u_t) = f(u_{t+1})$, where

$f^2(u_t) := r[ru_t\,(1-u_t)][1 - ru_t\,(1-u_t)]$.

The steady states for this equation satisfy:

$u_s = r[ru_s\,(1-u_s)][1 - ru_s\,(1-u_s)]$.

. This is a quartic. But, we know two solutions: $u_s = 0, \frac{r-1}{r}$. So, we can factorize and we are left with a quadratic, whose solution is:

$$u_s^{\pm} = \frac{r+1}{2r} \pm \frac{1}{2r}\sqrt{(r-1)^2 - 4}. \tag{2.12}$$

These roots are real if $(r-1)^2 \geq 4$, *i.e.* $r \geq 3$.

These are the values of u_t that emerge from the pitchfork bifurcation at $r = 3$, with $f(u_s^+) = u_s^-$, and $f(u_s^-) = u_s^+$ (**Exercise**).

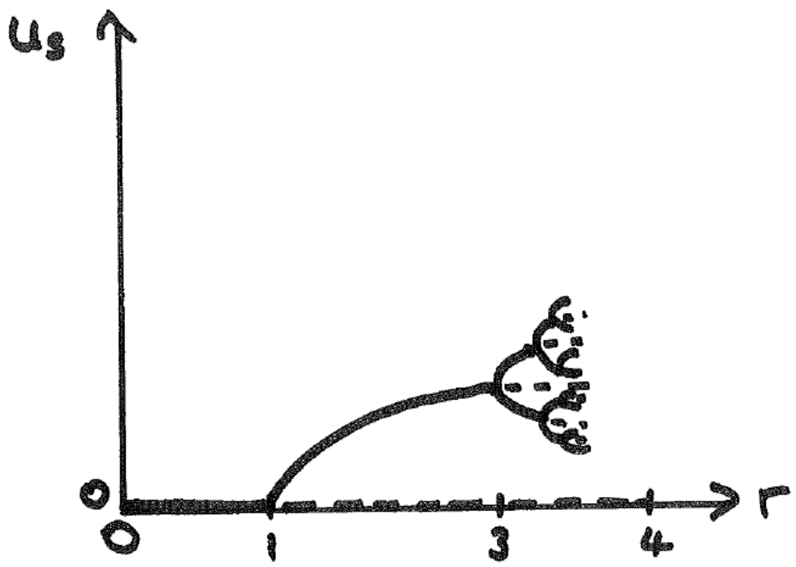
How do we find the linear stability of this periodic solution?

We determine the linear stability of the periodic solution in the same way as before: We define λ as

$$\begin{aligned}\lambda &= \frac{df^m(u)}{du}|_{u=u_i}, i = 0, \text{or } 1, \text{or } 2, ...\text{or m} - 1 \\ &= \frac{df(Q(u))}{du}|_{u=u_i} \\ &= f'(Q(u))\frac{dQ(u)}{du}|_{u=u_i} \\ &= f'(u_{i-1})\frac{df^{m-1}(u)}{du}|_{u=u_i}, \end{aligned} \tag{2.13}$$

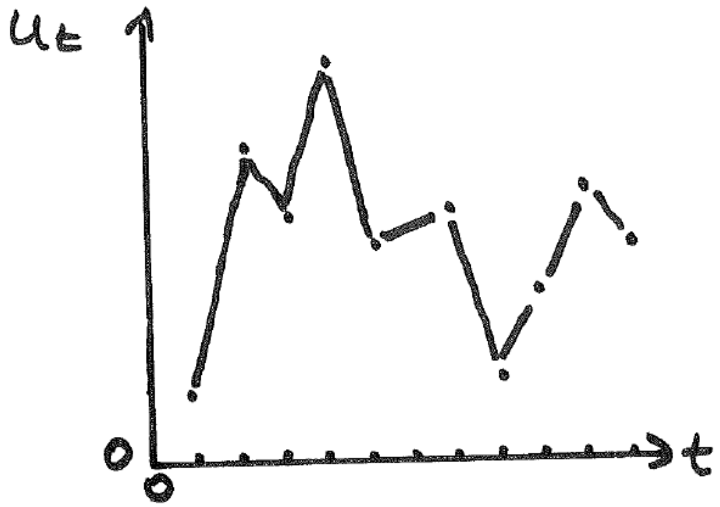
where $Q(u) = f^{m-1}(u)$. Hence, by iteration, we have that the state is linearly stable if

$$|\prod_{i=0}^{m-1} f'(u_i)| < 1. \tag{2.14}$$



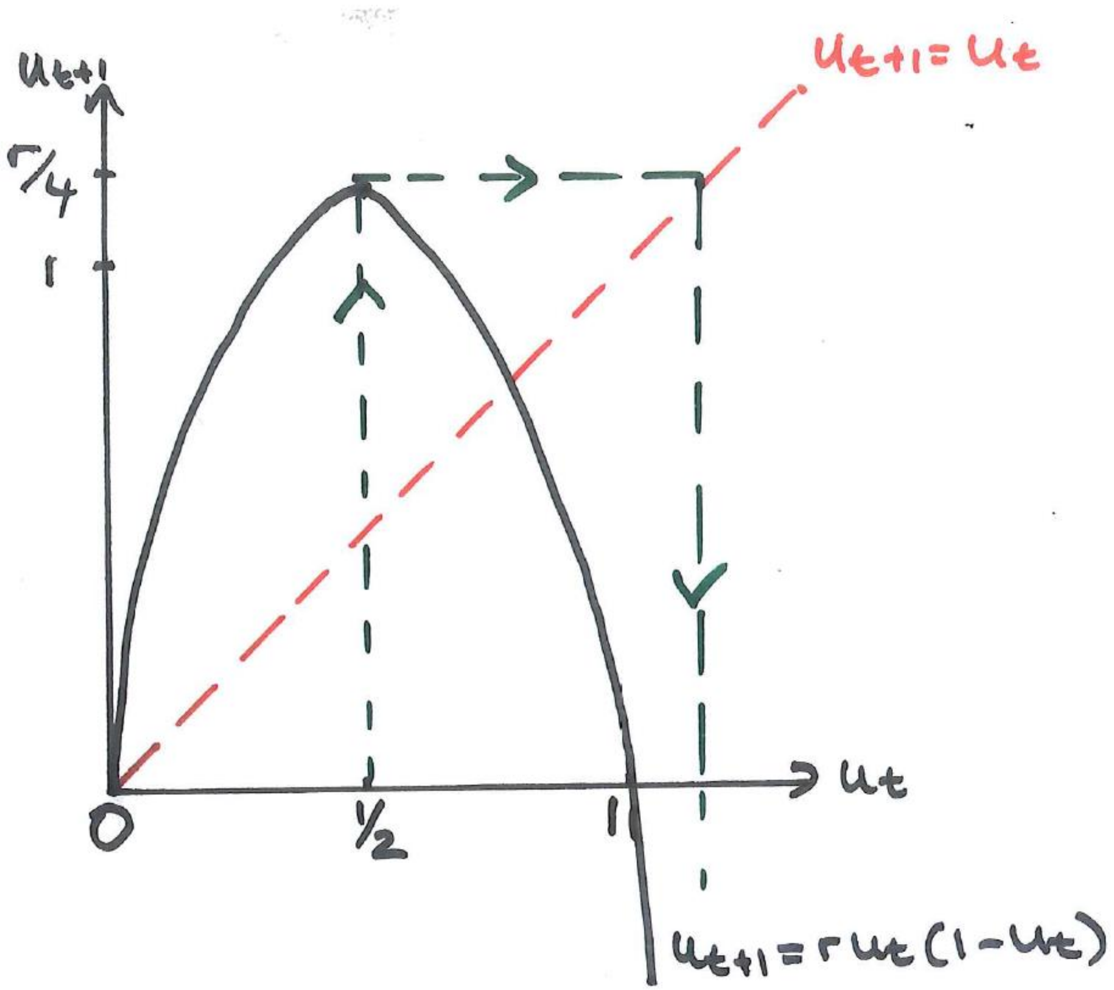
There exist a series r_c of values of r such that cycles bifurcate at these points with even orders. The limit of these points is called the Feigenbaum number (3.828...).

For r greater than this limit point, but less than 4, we have a period 3 oscillation. This implies *chaos* (Yorke and Li, 1975).



u_t
0
0
t

Note that for $r > 4$, this model is not realistic.



Summary

- Periodic solutions
- Chaos
- The discrete logistic growth model is *very* different its continuous counterpart!

End of Lecture 2_2